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COMPARSION OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract

In this contribution we compare solutions of second order linear differential equations with constant coefficients with respect to the form of right-hand side of the equation and to the form of initial or boundary conditions. We will see that some types of such differential equations one can resolve by the method of variation of constants and it is not possible solve them by Laplace transform. On the other hand, some of them are solvable by Laplace transform and are unsolvable by the method of variation of constants. The reason for the comparison is to show that the students of automation have to know both ways of solving linear differential equations with constant coefficients.

Key words

linear differential equation, Laplace transform, variation of constants

Introduction

Consider the following problem: solve linear differential equation of second order

$$y''(t) + a_1 y'(t) + a_2 y(t) = g(t), \quad a_1, a_2 \in \Re \tag{1}$$

with either initial conditions $y(t_0) = a, y'(t_0) = b$ or with boundary conditions of the form $y(t_0) = a, y(t_1) = b$ or $y(t_0) = a, y'(t_1) = b$ or $y'(t_0) = a, y'(t_1) = b$. The right-hand side of (1) - the function $g(t)$ - can by continuous as well as discontinuous on $(0;\infty)$. It is well

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known, that one can solve the differential equation by the method of variation of constants only if the function $g(t)$ is continuous (and it is impossible to use the method whenever $g(t)$ is discontinuous). On the other hand, one can solve the equation with the help of Laplace transform only when the initial conditions are given (and it is not possible to use the method when boundary conditions are given). In the next part, we show the examples of solving the differential equation in various cases.

Problem solution

Problem 1: Solve the differential equation $y'' + 4\mathrm{y} = \mathrm{t}^2 + 1,$ with boundary conditions $y(0) = 1,\ \ y'\left(\frac{\pi}{2}\right) = 0.$

Solution: The equation is an inhomogeneous linear differential equation of second order with constant coefficients. Because $0 \neq \frac{\pi}{2}$, we cannot solve the equation with the help of Laplace transform. Firstly, we solve the homogeneous differential equation $y'' + 4\mathrm{y} = 0$ to obtain the general solution (complementary function) of the homogeneous differential equation. The characteristic equation is $\lambda^2 + 4 = 0,$ so $\lambda_1 = 2i, \lambda_2 = -2i.$ It follows that the two linearly independent solutions of the homogeneous differential equation are $y_1 = \cos 2t$ and $\mathrm{y}_2 = \sin 2t$ and so complementary function is of the form $y_c = c_1 y_1 + c_2 y_2 = c_1 \cos 2t + c_2 \sin 2t.$

The right-hand side of the original equation has a special form $g(t) = t^2 + 1$, so we can use the method of undetermined coefficients. Because the function $g(t)$ is a polynomial of degree two and zero is not a root of the characteristic equation, the particular solution of the inhomogeneous equation will be of the form $y_p = at^2 + bt + c$ with undetermined coefficients $a, b, c.$ Substituting $y_p = at^2 + bt + c$ and $y_p'' = 2a$ into the original equation we obtain $a = \frac{1}{4}, b = 0, c = \frac{1}{8}.$ The general solution of the inhomogeneous equation is the sum of the complementary function and the particular solution. So

$$y_g = y_c + y_p = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4}t^2 + \frac{1}{8},$$

where c_1 and c_2 are arbitrary real numbers. To obtain the coefficients c_1 and c_2 (for given boundary conditions) we have to derivate the function y_g:

$$y_g' = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{1}{2}t.$$

Substituting $y(0)=1$ and $y'\left(\frac{\pi}{2}\right)=0$ onto $y_g(t)$ and $y'_g(t)$, we get $c_1=\frac{7}{8}$, $c_2=\frac{\pi}{8}$ and the solution of the equation with boundary conditions on $\left\langle 0;\frac{\pi}{2}\right\rangle$ is the function $y(t)=\frac{7}{8}\cos 2t+\frac{\pi}{8}\sin 2t+\frac{1}{4}t^2+\frac{1}{8}$.

Problem 2: Solve the differential equation $y''-3y'+2y=g(t)$, where the function $g(t)$ is given by: $g(t)\begin{cases}0, \text{ for } t\in(-\infty;0)\bigcup(1;\infty)\\ 1, \text{ for } t\in\langle 0;1\rangle\end{cases}$, with the initial conditions $y(0)=0, y'(0)=1$.

Solution: Because the function is not continuous on $(0;\infty)$, we cannot use the method of variations of constants. Instead, we have to solve the equation with Laplace transform. For the originals y'', y', y and $g(t)$ we obtain the following images: $\mathcal{L}[y(t)]=Y(p)$, $\mathcal{L}[y'(t)]=p\mathcal{L}[y(t)]-y(0)=pY(p)$, $\mathcal{L}[y'']=p^2\mathcal{L}[y(t)]-py(0)-y'(0)=p^2Y(p)-1$ and $\mathcal{L}[g(t)]=\int_0^\infty g(t)e^{-pt}dt=\int_0^1 1\cdot e^{-pt}dt+\int^\infty 0\cdot e^{-pt}dt=\int_0^1 e^{-pt}dt=-\frac{1}{p}\left[e^{-pt}\right]_0^1=\frac{1}{p}\left(1-e^{-p}\right)$.

Substituting onto the differential equation we get $\left(p^2Y(p)-1\right)-3pY(p)+2Y(p)=\frac{1}{p}\left(1-e^{-p}\right)$, from which we obtain the image of the solution $Y(p)=\frac{p+1}{p(p-1)(p-2)}-\frac{e^{-p}}{p(p-1)(p-2)}$.

After partial fraction decomposition, we will have $Y(p)=\frac{1}{2}\cdot\frac{1}{p}-2\cdot\frac{1}{p-1}+\frac{3}{2}\cdot\frac{1}{p-2}-\frac{1}{2}\cdot\frac{e^{-p}}{p}+\frac{e^{-p}}{p-1}-\frac{1}{2}\cdot\frac{e^{-p}}{p-2}$. The solution of the differential equation $y(t)$ is the original of $Y(p)$. So, for $y(t)$ we have $y(t)=\mathcal{L}^{-1}[Y(p)]=\frac{1}{2}-2e^t+\frac{3}{2}e^{2t}-\frac{1}{2}\eta(t-1)+e^{t-1}\eta(t-1)-\frac{1}{2}e^{2(t-1)}\eta(t-1)$ or equivalently

$$y(t)=\begin{cases}\frac{1}{2}-2e^t+\frac{3}{2}e^{2t}, \text{ for t}\in(-\infty;1)\\ \left(\frac{1}{e}-2\right)e^t+\left(\frac{3}{2}-\frac{1}{2e^2}\right)e^{2t}, \text{ for } t\in\langle 1;\infty)\end{cases}.$$

Problem 3: Solve the differential equation $y''-3y'+2y=t^2+1$, with initial conditions $y(0)=0$, $y'(0)=0$.

Solution: The equation is an inhomogeneous linear differential equation of second order with constant coefficients. Because the right-hand side of the equation is continuous and, additionally, it is of a special type, we can solve the equation with the Laplace transform as well as by the method of undetermined coefficients.

a) Method of undetermined coefficients: We solve the homogeneous differential equation $y'' - 3y' + 2y = 0$. The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$ with the roots $\lambda_1 = 1$ and $\lambda_2 = 2$. The two linearly independent solutions of the homogeneous differential equation are $y_1 = e^t$ and $y_2 = e^{2t}$. The complementary function is $y_c = c_1 y_1 + c_2 y_2 = c_1 e^t + c_2 e^{2t}$.
The right-hand side of the original equation has a special form $g(t) = t^2 + 1$, so we can use the method of undetermined coefficients. As in the problem 1, the right-hand side of the differential equation is a polynomial of degree two and zero is not a root of the characteristic equation. So, the particular solution of the inhomogeneous equation will be again of the form $y_p = at^2 + bt + c$ with undetermined coefficients a, b, c. Substituting $y_p = at^2 + bt + c$, $y_p' = 2at + b$ and $y_p'' = 2a$ into the original equation we obtain $a = \frac{1}{2}, b = \frac{3}{2}, c = \frac{9}{4}$. The general solution is the inhomogeneous equation $y_g = y_c + y_p = c_1 e^t + c_2{}^{2t} + \frac{1}{2}t^2 + \frac{3}{2}t + \frac{9}{4}$, c_1 and c_2 in $\Re$. Substituting the initial conditions $y(0) = 0$, $y'(0) = 0$ into $y_g(t)$ and $y_g'(t) = c_1 e^t + 2c_2 e^{2t} + t + \frac{3}{2}$ we obtain the solution of the differential equation with initial conditions $y(t) = -3e^t + \frac{3}{4}e^{2t} + \frac{1}{2}t^2 + \frac{3}{2}t + \frac{9}{4}$.

b) Laplace transform: For the originals y'', y', y and $t^2 + 1$ we obtain the images: $\mathcal{L}[y(t)] = Y(p)$, $\mathcal{L}[y'(t)] = p\mathcal{L}[y(t)] - y(0) = pY(p)$, $\mathcal{L}[y''] = p^2\mathcal{L}[y(t)] - py(0) - y'(0) = p^2 Y(p)$ and $\mathcal{L}[t^2 + 1] = \frac{2}{p^3} + \frac{1}{p}$. Substituting onto the differential equation we get $p^2 Y(p) - 3pY(p) + 2Y(p) = \frac{2}{p^3} + \frac{1}{p}$. So, the image of the solution $Y(p) = \frac{p^2 + 2}{p^3 (p-1)(p-2)}$. After partial fraction decomposition we get the image in the form $Y(p) = \frac{9}{4} \cdot \frac{1}{p} + \frac{3}{2} \cdot \frac{1}{p^2} + \frac{1}{p^3} - 3 \cdot \frac{1}{p-1} + \frac{3}{4} \cdot \frac{1}{p-2}$. The solution of the differential equation $y(t)$ is the original of $Y(p)$. So, for $y(t)$ we have $y(t) = \mathcal{L}^{-1}[Y(p)] = \frac{9}{4} + \frac{3}{2} \cdot \frac{t^1}{1!} + \frac{t^2}{2!} - 3e^t + \frac{3}{4}e^{2t} = -3e^t + \frac{3}{4}e^{2t} + \frac{1}{2}t^2 + \frac{3}{2}t + \frac{9}{4}$.

Results and discussions

In the previous section we have seen that the methods of solution of second order linear differential equation depend on the form of the right-hand side and on the types of conditions setting on solutions. We showed solutions in some particular cases.

Conclusion

Dynamical systems (which are one of main objects in the area of operations research**)**, can be described by differential equations of various orders. Frequently, such a differential equation is an $n-$ th order homogeneous linear differential equation with constant coefficients. Looking for a reaction of the system on input function, students of automation often have to solve such (inhomogeneous) equations. In regard to the form of the right-hand side (input function) and on the type of starting conditions they have to know both ways of solving linear differential equations with constant coefficients – the method of variation of constants and Laplace transform. In this contribution, we showed how such equation can look like and how to solve it.

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