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DETERMINATION OF MODULUS OF ELASTICITY AND SHEAR MODULUS BY THE MEASUREMENT OF RELATIVE STRAINS

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Abstract

This contribution is focused on determining the material properties (Young modulus and shear modulus) of the testing samples. The theoretical basis for determining material properties are the knowledge of linear elasticity and strength. The starting points are dependencies among the modulus of elasticity, shear modulus, normal stress and relative strain. The relative strains of the testing samples were obtained by measuring predefined load conditions using a strain-gauge bridge and the universal measurement system Quantum X MX 840. The integration of these tasks into the teaching process enhances practical and intellectual skills of students at secondary level technical universities.

Key words

Young modulus, shear modulus, torsion, simple bending, stress in the cross-section, relative strain

INTRODUCTION

Both the Young modulus and shear modulus are determined experimentally. The selection of the method depends on the type of load (static, dynamic), on the geometry of test samples and also on the characteristics of the investigated material, which we need to identify.

A Stress - Strain diagram may be used to determine the Young modulus by tensile test. In this case, samples of circular cross sections are used (1). The method of holographic interferometry can be applied to determine the Young modulus for samples of irregular shape (2). It is also possible to use e.g. the pulse method for determining the resonance frequency (3). Except from the previous method, the shear modulus can be determined also by the torsional pendulum method (4).

This article describes the determination of the Young modulus on the basis of a bending rod with rectangular cross-section. A rod with a circular cross-section loaded by torque was used for the determination of the shear modulus. In both cases, strains of samples were measured by strain-gauges.

THEORETICAL BACKGROUND

Young modulus is defined by Hooke`s law (5):

$$E = \frac{\sigma}{\varepsilon}, \qquad [1]$$

where E [Pa] is the Young modulus, σ [Pa] is the tensile (normal) stress, ε [-] is the relative strain.

The rod, fixed on one side and loaded by the force on the other side is shown in Fig. 1. The normal stress σ_{max} has the maximum value at the ends of the fiber of the cross-section (Fig. 2):

$$\sigma_{max} = \frac{M_0(x)}{J_z} y_{max} = \frac{M_0(x)}{W_0} = \frac{x\,F}{W_0}, \qquad [2]$$

where J_z [m^4] is the second moment of area, W_0 [m^3] is the section modulus in the bending, $M_0(x)$ [N·m] is the bending moment, F [N] is the loading force, x [m] is the distance from the unbridled end of the rod.

For a rectangular cross-section, the following equations should be applied

$$J_z = \frac{b\,h^3}{12},\ y_{max} = \frac{h}{2} \text{ and } W_0 = \frac{J_z}{y_{max}} = \frac{b\,h^2}{6}.$$

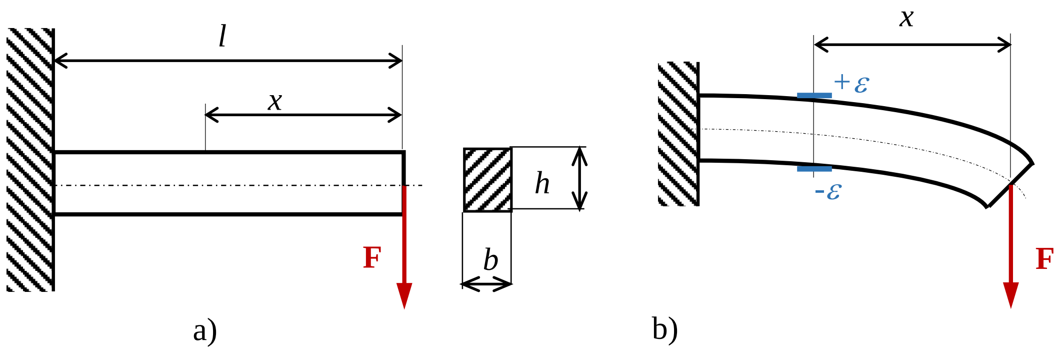


***Fig. 1** Loading and fixing of the testing rod a) undeformed shape; b) deformed shape*

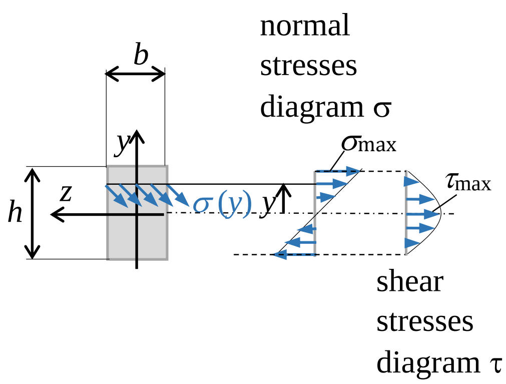


***Fig. 2** The normal and shear stress diagrams in any rectangular cross-section of a rod*

When the rod is under simple bending, the shear stress is equal to zero in the edge fibers of the cross-section. If the angle strain equals to zero in the cross-section, only relative strain appears there. The magnitude of relative strain changes as section height does. The greatest prolongation ($+\varepsilon$) and the greatest shortening ($-\varepsilon$) appears in extreme fibers respectively, where the greatest normal stress is observed (6). If we are able to measure the relative strain, we will calculate the modulus of elasticity E on the base of the relationships [1] and [2]

$$E = \frac{\sigma_{\max}}{\varepsilon} = \frac{F \cdot x}{\frac{b \cdot h^2}{6} \cdot \varepsilon} = \frac{6F}{b \cdot h^2} \frac{x}{\varepsilon}\,. \qquad [3]$$

The value of the shear modulus G was determined on the basis of stress in rods with a circular cross-section under torsion moment (Fig. 3a). In practice, solid bodies with circular (hollow circular) cross-section are quite often used in the case of twisting moment loading. If the rod is loaded by a simple twist, the individual cross-sections of the rod are stressed by shear stress. The maximum shear stress is at the edge fibers of the cross-section (Fig. 3b) and it is given by (7)

$$\tau_{\max} = \rho_{\max} \frac{M_k}{J_p} = \frac{M_k}{\frac{J_p}{\rho_{\max}}} = \frac{M_k}{\frac{J_p}{r}} = \frac{M_k}{W_k}\,, \qquad [4]$$

where ρ [m] is the distance between the fiber and the axis of cross-section of the shaft ($\rho_{\max} = r$), r [m] is the radius of the circular cross-section, M_k [N·m] is the torsion moment, J_p [m^4] is the polar moment of inertia of the cross-section, W_k [m^3] is the torsional modulus in the cross-section.

For circular cross-section: $J_p = \frac{\pi\, d^4}{32}$, $d = 2r$ and $W_k = \frac{\pi\, d^3}{16}$.

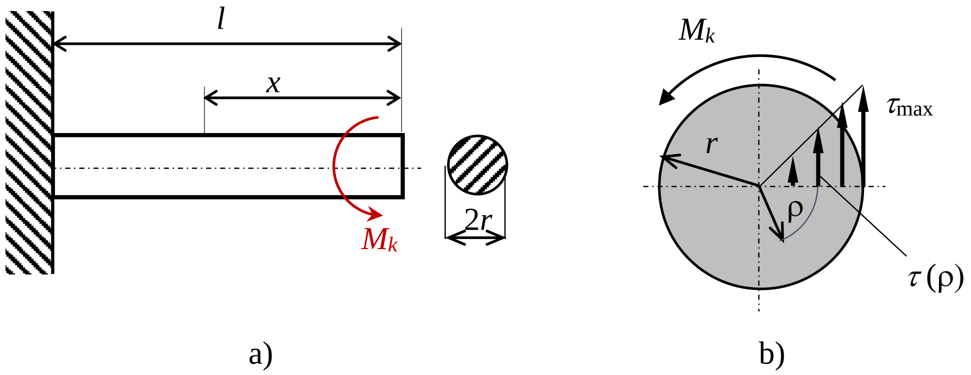


Fig. 3 *The rod of the circular cross-section loaded by simple torsion moment*
a) fixing and loading of the rod (shaft)
b) the shear stresses diagram in any circular cross-section of the rod (shaft)

On the basis of the equations of general Hooke's law for planar stresses and considering the pure shear (Fig. 4) we get (8)

$$\tau_{\max} = |\sigma_1| = |\sigma_2| = +\frac{E\varepsilon}{1+\mu}\,, \qquad [5]$$

where σ_1, σ_2 [Pa] are the principal stresses ($\sigma_2 = -\sigma_1$), E [Pa] is the Young modulus, μ [-] is the Poisson` ratio, ε_1, ε_2 [-] are the principal strains ($\varepsilon = \varepsilon_1$, $\varepsilon_2 = -\varepsilon_1$).

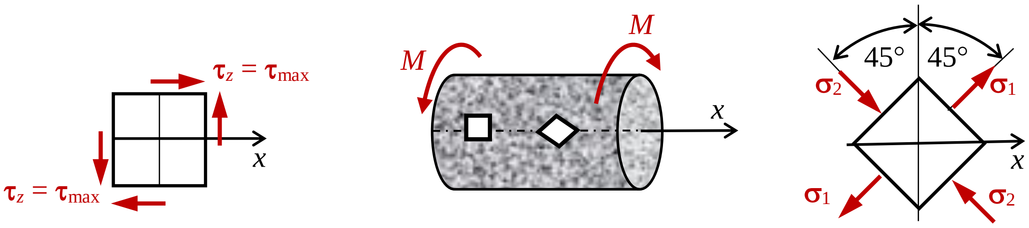


Fig. 4 *Stresses by pure shear loading*

The shear modulus is (6):

$$G = \frac{E}{2(1+\mu)}\,, \quad [6]$$

respectively by modifying the equations [4 - 6] we get:

$$G = \frac{\tau_{\max}}{2\varepsilon} = \frac{\frac{M_k}{W_k}}{2\varepsilon}. \quad [7]$$

THE STRAIN-GAUGE MEASUREMENT OF RELATIVE STRAINS

Figure 5a shows the schemas of the measuring apparatuses applied for determination of the dependence of the relative strains on the applied load. In both cases, one end of the bar was rigidly attached to the metal frame. Loading force was created by the weight of a known mass. The ballast was hanged at the free end of the bar when loading under bending. When loading by torsion moment, the ballast was hanged on the arm as illustrated in Fig. 5b. Strain-gauges were standardly glued to the surface of the test samples.

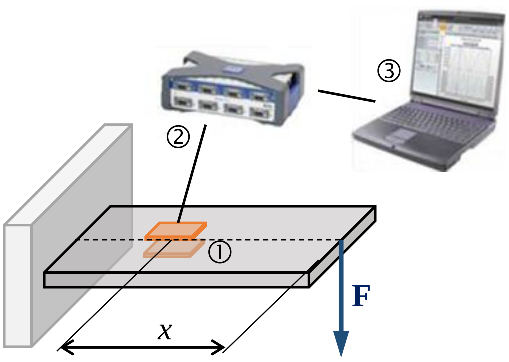
At loading under bending: Strain gauges were glued to the surface of the sample (one to the upside and second to the underside of the rod) at a distance *l* from the free end. The axes of strain gauges were parallel to the longitudinal axis of the rod. In this case, measuring strain gauges were loaded by maximum normal stress and they were able to record the relative strains on the surface of the rod.

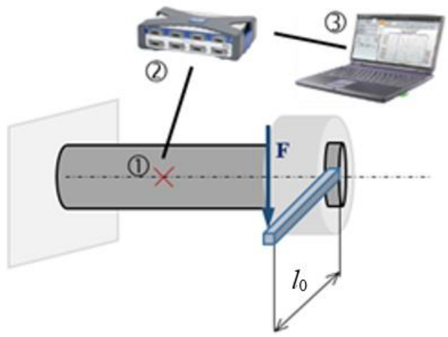
Strain-gauges cross with two perpendicular axes were used in case of the sample loaded under twist. Strain-gauges sensors were attached to the surface of the test samples so that axes of the strain-gauges and the longitudinal axis of the sample form the angles of 45° and 135°. In this case, measuring strain-gauges were stressed with pure shear (Fig. 4). The axes of the strain-gauges were consistent with the directions of the axes of the principal stresses and they were able to record the principal relative strains.

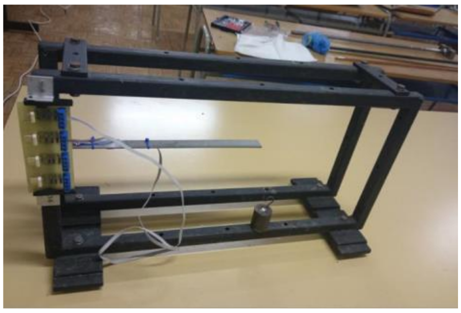
Relative change in resistance is recorded by the strain gauges. This change is directly proportional to the relative change of the active length of strain-gauges

$$\frac{\Delta R}{R} = k\frac{\Delta l}{l} = k\varepsilon \quad [8]$$

where ΔR [Ω] is the resistance change of strain-gauge, R [Ω] is the resistance of the strain-gauge before deformation, k [-] is the gauge factor (given by the manufacturer), Δl [m] is the change of the active length of the strain gauge, l [m] is the active length of the strain gauge, ε is the relative strain.







a) b)

① strain-gauges ② universal measurement system Quantum X MX 840
③ control computer

Fig. 5 *The scheme and photos of measurement apparatus*
a) The sample loaded under bending
b) The sample loaded under twist

Values of resistance, which are mentioned to be recorded by strain-gauge sensor, achieved values of order from 10^{-4} Ω to 10^{0} Ω. Therefore, involvement of the strain gauges in the Wheatstone bridge is used. The equation that can be used in practice to determine the strain has the form (9):

$$\frac{\Delta U_0}{U_n} = \frac{1}{4}\left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} + \frac{\Delta R_3}{R_3} - \frac{\Delta R_4}{R_4}\right) = \frac{1}{4}k(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4), \quad [9]$$

where k is the gauge factor, ε_i are the individual relative strains, R_i are the individual resistance, ΔU_0 [μV] is change of the output voltage, U_n [μV] is the driving voltage.

Two strain gauges were involved in a half bridge in these measurements. This means, the resistors R_1 and R_2 were replaced by measuring strain gauges in the Wheatstone bridge. Thus, for a relative strains, the following applies: $\varepsilon_1 = -\varepsilon_2 \neq 0$ a $\varepsilon_3 = \varepsilon_4 = 0$. Substituting these values into [9] we get

$$\frac{\Delta U_0}{U_n} = \frac{1}{4}k(\varepsilon_1 - \varepsilon_2) = 2\frac{1}{4}k\varepsilon_1. \quad [10]$$

Connection of the sensors with a measuring system was performed using Ethernet cables. The HBM Catman Easy program (10) was used to record data, manage the measurement system and to process the obtained data.

DETERMINATION OF THE VALUES OF YOUNG MODULUS AND SHEAR MODULUS

Table 1 shows the measured values of the parameters necessary to determine the Young modulus (Eq. 3, Fig. 5a) and used in the tests.

Strain-gauge transducers 6JP with an ohmic value of 117 Ω and with the gauge factor k = 2.01 were used to measure relative strains. Six measurements were performed using the measurement system QuantumX MX840 [10]. These measurements determined the final value of ε_1 = 0.00020517 m.m^{-1}. All measured values were substituted into the relationship [3]:

$$E = \frac{6F}{b \cdot h^2}\frac{l}{\varepsilon} = 2.088 \cdot 10^{11}\text{Pa} = 2.088 \cdot 10^{5}\ \text{MPa}.$$

Table 1 Values of the parameters for the determination of Young modulus

F = 1660.54 ·10^{-3} N	magnitude of the loading force F (F = m·g; g = 9.81 m s^{-2})
m = 169.27· 10^{-3} kg	the mass weight
b = 19.55· 10^{-3} m	width of the cross-section of the sample
h = 1.75· 10^{-3} m	thickness of the cross-section of the sample
l = 257.50· 10^{-3} m	distance between the strain-gauges and the line of the force

Torque (M_k) was triggered by weights of different masses (m) which were strapped to the end of the arm of the length (l_0):

$$M_k = F \cdot l_0 = m \cdot g \cdot l_0. \quad [11]$$

Table 2 summarizes the measured and the calculated values of the parameters which are necessary to determine the shear modulus (Eq. 7, Fig. 5b) and which were used in the tests.

Table 2 Parameters of the bar with circular cross-section

l_0 = 0.5 m	length of the arm
d = 16.18 · 10^{-3} m	diameter of the test bar
W_k = 831.7· 10^{-9} m^3	torsional modulus

Strain gauge transducers KM 120 with an ohmic value of 120 Ω and with the gauge factor 2.06 were used for measurement. Table 3 shows the values of the relative strains for the individual loading torsional moments obtained using the measurement system QuantumX MX840. There are also calculated values of the shear modulus for the individual relative strains on the base of relations [7] and [11] in the table.

Table 3 Values of the parameters for the determination of shear modulus

m ·10^{-3} [kg]	M_k ·10^{-3} [N·m]	ε ·10^{-5} [–]	G ·10^{6} [Pa]
998.83	4899.26	3.46	85 199.00
2037.63	9994.58	7.05	85 203.09
3076.57	15090.58	10.65	85 152.31
4105.02	20135.12	14.20	85 269.08
5143.96	25231.12	17.95	84 484.82
6178.91	30307.55	21.34	85 400.72
7217.62	35402.43	25.06	84 925.46

The final value of the shear modulus was determined by the average of the measured values. Determining of the value of the uncertainty on the shear modulus (δG) was determined by the relationship:

$$\delta G = \sqrt{\frac{\sum(G_i - \bar{G})^2}{n(n-1)}}, \quad [12]$$

where $\bar{G}$ is the mean value of the measured values, n is the number of measurement, G_i is the size G in the i^{th} measurement.

We obtained the following values of shear modulus $G = (85\,090.64 \pm 114.48) \cdot 10^6$Pa. The relative error is 0.14 %.

CONCLUSION

This contribution deals with determination of the modulus of elasticity for a sample with rectangular cross-section and shear modulus for rods with circular cross-sections. Each method used should be classified as static methods. Relative strains were detected experimentally for the given load force (bending) and for the individual values of the loading torque (twist). Values of modulus were calculated by relations of the elementary elasticity and values obtained by measurements. Tenzometric sensors connected to the half bridge and universal measurement system QuantumX MX840 were used to experimentally obtain the values of relative deformations.

The supplier of the test sample declared that the magnitude of the modulus of elasticity is $2.1 \cdot 10^{11}$ Pa. The value determined for the Young modulus by test was $2.088.10^{11}$ Pa. The relative error of determining the Young modulus applying this method was 0.57 %. The mechanical table declares the value of the modulus of elasticity for steel within the range $(1.9$ to $2.15) \cdot 10^{11}$ Pa.

For a steel sample with circular cross-section we obtained the value of the shear modulus 85 090.64 MPa with the relative error of 0.14 %. According to the standards, steel has the value of the shear modulus within the range of 80 000 MPa to 85 0000 MPa. In both cases, the obtained values are sufficiently accurate. So it can be concluded that the applied method is suitable to include into the teaching process at technical universities. Thus, the students should have the opportunity to see the link between theory and practice.

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