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THE VERIFICATION OF INFLUENCE OF THE POINT „C“ POSITION FROM GIVEN INTERVAL TO SOLVING SYSTEMS WITH HIGH-SPEED FEEDBACK

Ingrida BAJČIČÁKOVÁ, Dominika JUROVATÁ

Mgr. Ingrida Bajčičáková, PhD., Ing. Dominika Jurovatá, PhD., Slovak University of Technology in Bratislava, Faculty of Materials Science and Technology in Trnava, Advanced Technologies Research Institute, Bottova 25, 917 24 Trnava, Slovakia
e-mail: ingrida.bajcicakova@stuba.sk, dominika.jurovata@stuba.sk

Abstract

This article deals with the design of effective numerical scheme for solving three point boundary value problems for second-order nonlinear singularly perturbed differential equations with initial conditions. Especially, it is focused on the analysis of the solutions when the point c from given interval is not the centre of this interval. The obtained system of nonlinear algebraic equations is solved by Newthon-Raphson method in MATLAB. It also verifies the convergence of approximate solutions of an original problem to the solution of reduced problem. We discuss the solution of a given problem with the situation when the point c is in the middle of the given interval.

Key words

Nonlinear dynamic systems, MATLAB, high-speed feedback, numerical scheme

INTRODUCTION

Nonlinear second-order dynamic systems with high-speed feedback are nowadays more and more popular as a result of their utilization in high frequency oscillators in electronic circuits and in diffusion processes control. The issue is therefore topical not only for the mathematicians' community who deal with the theory and application of nonlinear dynamic systems, but it is important also for the field of automation. The fact that the differential equation investigation with small parameters at the highest derivation has been paid such attention is the result of the fact that they represent a suitable model for the description of systems with strong nonlinearities and high frequency circuits (e.g. Duffing oscillators).

SOLVING THE NONLINEAR EQUATION SYSTEMS

Whereas for one nonlinear equation, we have a sufficient amount of reliable methods for separation and precision, for the systems of nonlinear equations the situation is just the opposite. For systems with two or three equations, we can carry out the separation of roots at least graphically. In the case of more equations, the possibility does not exist. We have several methods to make the systems´ roots more precise; however, their efficiency depends on the accuracy of the initial approximation (5).

Simple-iteration method

Let us use the system n of equations with n unknown in the form of:

$$\begin{gathered} f_1(x_1, x_2, \dots, x_n) = 0 \\ f_2(x_1, x_2, \dots, x_n) = 0 \\ \dots \\ f_n(x_1, x_2, \dots, x_n) = 0. \end{gathered} \quad [1]$$

The system can be modified to the form:

$$\begin{gathered} g_1(x_1, x_2, \dots, x_n) = x_1 \\ g_2(x_1, x_2, \dots, x_n) = x_2 \\ \dots \\ g_n(x_1, x_2, \dots, x_n) = x_n, \end{gathered}$$

or to the vector form: $x = G(x)$, where $G = (g_1, \dots, g_n)^T$. Now, we select the initial approximation $x^{(0)}$ and calculate the sequence of the progressive approximations from the iteration relation:

$$x^{(k+1)} = G\left(x^{(k)}\right).$$

As we operate with n functions with n unknowns, via the derivation G' the matrix will be in the form:

$$G' = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \dots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \dots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots \quad \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} \quad \dots & \frac{\partial g_n}{\partial x_n} \end{pmatrix}.$$

If the functions $g_1, \dots, g_n$ are differentiable, then the condition of convergence exists for the method of simple iteration.

If G illustrates a closed area and it is differentiable within, and if there is such a number as $\alpha \in \langle 0, 1)$, that $\|G'\| \leq \alpha$ for all x in this area, then in this area there is a fixed point δ of G images. Then the sequence of the progressive approximations to this point converges for a random initial approximation $x^{(0)}$ of the selected area. $\|G'\|$ is of a row or column character, or the column norm of the matrix G'.

As the verification of the convergence condition can be difficult, it is appropriate to determine the maximum number of method steps, and if they are exceeded, to accomplish the calculation with the result that the method diverges. Then it is necessary to select another initial approximation, a new iteration function, or a new method. Finding the suitable iteration function can be very difficult (6).

Jacobi method

The Jacobi method for solving the system of nonlinear equations [1] is similar to the Jacobi iteration method for solving the system of linear equations. In general, from the i-th equation for $k + 1$ iteration, the i-th unknown x_i (i.e. $x_i^{(k+1)}$) is expressed , while into other parameters $(x_1, x_2, \dots x_{i-1}, x_{i+1}, \dots, x_n)$ the values calculated in the previous iteration k, are plugged, i.e.:

$$\left(x_1^{(k)}, x_2^{(k)}, \dots x_{i-1}^{(k+1)}, x_{i+1}^{(k+1)}, \dots, x_n^{(k)}\right).$$

If we have the system of nonlinear equations, then, according to their number (and hence the number of unknowns), the Jacobi form is modified as well. Its general form is as follows:

$$J\left(x^{(k)}\right) = \begin{bmatrix} \frac{\partial f_1^{(k)}}{\partial x_1} & \frac{\partial f_1^{(k)}}{\partial x_2} & \dots & \frac{\partial f_1^{(k)}}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n^{(k)}}{\partial x_1} & \frac{\partial f_n^{(k)}}{\partial x_2} & \dots & \frac{\partial f_n^{(k)}}{\partial x_n} \end{bmatrix},$$

where the indication $\partial f_i^{(k)} / \partial x_j$ generally represents the expression (4):

$$\frac{\partial f_i^{(k)}}{\partial x_j} = \frac{\partial f_i\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_j}.$$

Newton-Raphson method

This method for solving the system of nonlinear equations represents the direct enhancement of the Newton method for solving one nonlinear equation. If we state the system of nonlinear equations in the form of:

$$\begin{gathered} f_1(x_1, x_2, \dots, x_n) = 0 \\ f_2(x_1, x_2, \dots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) = 0. \end{gathered}$$

Subsequently, we can develop these functions into Taylor order to the point $x^{(k)} = \left(x_1^k, x_2^k, \dots, x_n^k\right)$, where k represents k-th iteration and hence $x^{(k)}$ is the result of the k-th iteration.

The breakdown then looks as follows:

$$f_1(x_1, x_2, \dots, x_n) = f_1\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right) + \frac{\partial f_1\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_1}\left(x_1 - x_1^{(k)}\right) + \frac{\partial f_1\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_2}\left(x_2 - x_2^{(k)}\right) + \cdots + \frac{\partial f_1\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_n}\left(x_n - x_n^{(k)}\right) + o(x^2) = 0$$

$$f_2(x_1, x_2, \dots, x_n) = f_2\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right) + \frac{\partial f_2\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_1}\left(x_1 - x_1^{(k)}\right) + \frac{\partial f_2\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_2}\left(x_2 - x_2^{(k)}\right) + \cdots + \frac{\partial f_2\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_n}\left(x_n - x_n^{(k)}\right) + o(x^2) = 0$$

....

$$f_n(x_1, x_2, \dots, x_n) = f_n\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right) + \frac{\partial f_n\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_1}\left(x_1 - x_1^{(k)}\right) + \frac{\partial f_n\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_2}\left(x_2 - x_2^{(k)}\right) + \cdots + \frac{\partial f_n\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_n}\left(x_n - x_n^{(k)}\right) + o(x^2) = 0$$

If we consider only linear members of this development, then we can write the system in the matrix form as follows:

$$-\begin{bmatrix} f_1\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right) \\ f_2\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right) \\ \dots \\ f_n\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\partial f_1\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_1} & \frac{\partial f_1\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_2} \dots \frac{\partial f_1\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_n} \\ \frac{\partial f_2\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_1} & \frac{\partial f_2\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_2} \dots \frac{\partial f_2\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_n} \\ \dots \\ \frac{\partial f_n\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_1} & \frac{\partial f_n\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_2} \dots \frac{\partial f_n\left(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\right)}{\partial x_n} \end{bmatrix}.$$

$$\begin{bmatrix} \left(x_1 - x_1^{(k)}\right) \\ \left(x_2 - x_2^{(k)}\right) \\ \dots \\ \left(x_n - x_n^{(k)}\right) \end{bmatrix}$$

or in the shortened form:

$$-f\left(x^{(k)}\right) = J\left(x^{(k)}\right)\left(x - x^k\right),$$

where the matrix $J\left(x^{(k)}\right)$ is called Jacobian (or Jacobi matrix). From this equation, the vector x can be expressed symbolically, and it can be replaced by the new iteration value of $x^{(k+1)}$ and we obtain the equation in the form of:

$$x^{(k+1)} = x^{(k)} - J\left(x^{(k)}\right)^{-1} f\left(x^{(k)}\right).$$

The Newton-Raphson method converges quadratically (4, 7).

Method of Steepest Descent

The Descent method is utilized to search for the function minimum and can also be used for solving the systems of nonlinear equations. If we state the system of nonlinear equations in the form of:

$$\begin{gathered} f_1(x_1, x_2, \dots, x_n) = 0 \\ f_2(x_1, x_2, \dots, x_n) = 0 \\ \dots \\ f_n(x_1, x_2, \dots, x_n) = 0, \end{gathered}$$

with the solution $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, if the function g is defined as:

$$g(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} [f_1(x_1, x_2, \dots, x_n)]^2,$$

with the minimum value equal to 0.

If we also state the initial estimation $x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix}$,

then it stands: $$x^{(1)} = x^{(0)} - \alpha \nabla g\left(x^{(0)}\right), \qquad [2]$$

for the constant $\alpha > 0$ and where $\nabla g(x^{(0)})$ is defined as:

$$\nabla g(x^{(0)}) = \left(\frac{\partial g}{\partial x_1}(x^{(0)}), \frac{\partial g}{\partial x_2}(x^{(0)}), \dots, \frac{\partial g}{\partial x_n}(x^{(0)}) \right)^T .$$

In the case that the curvature of the function is much too complex, then for the calculation of the local function minimum with the required precision, multiple iterations can be used.

The appropriate selection of α remains difficult, as it is necessary to ensure that $g(x^{(1)})$ is significantly smaller than $g(x^{(0)})$, hence, the relation $F(x_1^{(0)}) \leq F(x^{(0)})$ has to stand. The fixed selection of α, i.e. α will be the constant, and can result in inappropriate convergence. Therefore, to search for the suitable value of the variable α , we utilize the function of one variable in the form:

$$h(\alpha) = g\left(x^{(0)} - \alpha \nabla g(x^{(0)})\right).$$

The value of the variable α minimizing h is the searched value for the relation [2]. The method of finding the value is described in detail in the book Solving Nonlinear Equations Using Numerical Analysis (3) pages $4 - 7$.

This method converges to the solution only linearly, however, it will usually converge also for the unsuitably selected initial approximation (3). (1)

PROBLEM FORMULATION

To begin with, we again assume the nonlinear singularly perturbed system interpreted by DE in the form of:

$$\varepsilon y'' + ky = f(x, y), x \in \langle a, b \rangle, k < 0 \qquad [3]$$

with the boundary conditions:

$$y_a = y_c, \quad y_c = y_b, \quad a < c < b. \qquad [4]$$

We assume that point c is not the centre of the interval $\langle a, b \rangle$. The use of the numeric method y'' is replaced by the differential scheme and we modify the equation [3] to the general form:

$$\frac{\varepsilon}{h}\left(\frac{y_{i+1} - y_i}{h} - \frac{y_i - y_{i-1}}{h}\right) + ky_i = f(x_i, y_i), \qquad [5]$$

where k is a random constant and h is the selected step. Subsequently, we divide the interval $\langle a, b \rangle$ into N-intervals of the same length if c is not the centre of this interval. Then $i = 1, \dots, N - 1$ and N is the number of subintervals of the $\langle a, b \rangle$ (Fig. 1).

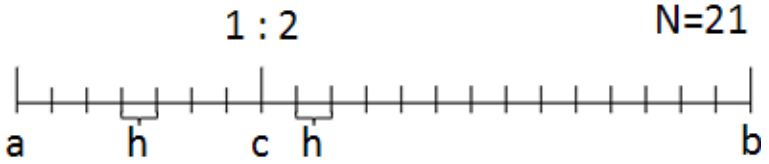

Fig. 1 *Example of division the interval $\langle a, b \rangle$ for N=21 if c is not the centre of this interval*

Thus for $i = 1,2, \dots, N-1$ on the interval $\langle a; b\rangle$ we obtain the function:

$$\begin{aligned} i = 1 \qquad & \frac{\varepsilon}{h}\left(\frac{y_2 - y_1}{h} - \frac{y_1 - y_0}{h}\right) + k y_1 = f(x_1, y_1), \\ i = 2 \qquad & \frac{\varepsilon}{h}\left(\frac{y_3 - y_2}{h} - \frac{y_2 - y_1}{h}\right) + k y_2 = f(x_2, y_2), \\ & \vdots \\ i = N-1 \qquad & \frac{\varepsilon}{h}\left(\frac{y_N - y_{N-1}}{h} - \frac{y_{N-1} - y_{N-2}}{h}\right) + k y_{N-1} = f(x_{N-1}, y_{N-1}), \end{aligned}$$

and the next two function from initial conditions [4]:

$$y_0 = y_{N/const}, \quad y_{N/const} = y_N.$$

Using this method we build the functions for the number of intervals given. In order for the boundary conditions to hold true, N must be an even number.

To calculate the system roots [3], [4], we use the Newton-Raphson iteration method, while owing to the demanding and time consuming solution, we utilize MATLAB program, in which we have created the program for solving the stated problem [3], [4] by the aforementioned method (1).

SOLUTION AND REACHED RESULTS IN *MATLAB*

MATLAB program was applied to the system [5], [4] for $N = 6$. Then interval $\langle 0; 1/2\rangle$ was divided into six subintervals $\langle 0, \frac{1}{12}; \frac{1}{6}; \frac{1}{4}; \frac{1}{3}; \frac{5}{12}; \frac{1}{2}\rangle$, giving variables $x_0, \dots, x_6$, and $c = \frac{1}{6}$. Functions for [5] were created, where $h = \frac{1}{12}, \varepsilon = \frac{1}{100}, k = -2$ and function $f(x, y) = y^2 + x$.

For $N = 6$, $i = 1, \dots, 5$, thus five functions were created:

$$\begin{aligned} f_1 &= \frac{3}{25} * \left(\frac{y_2 - y_1}{\frac{1}{12}} - \frac{y_1 - y_0}{\frac{1}{12}}\right) - 2 * y_1 - {y_1}^2 - \frac{1}{12}; \\ f_2 &= \frac{3}{25} * \left(\frac{y_3 - y_2}{\frac{1}{12}} - \frac{y_2 - y_1}{\frac{1}{12}}\right) - 2 * y_2 - {y_2}^2 - \frac{1}{6}; \\ f_3 &= \frac{3}{25} * \left(\frac{y_4 - y_3}{\frac{1}{12}} - \frac{y_3 - y_2}{\frac{1}{12}}\right) - 2 * y_3 - {y_3}^2 - \frac{1}{4}; \\ f_4 &= \frac{3}{25} * \left(\frac{y_5 - y_4}{\frac{1}{12}} - \frac{y_4 - y_3}{\frac{1}{12}}\right) - 2 * y_4 - {y_4}^2 - \frac{1}{3}; \\ f_5 &= \frac{3}{25} * \left(\frac{y_6 - y_5}{\frac{1}{12}} - \frac{y_5 - y_4}{\frac{1}{12}}\right) - 2 * y_5 - {y_5}^2 - \frac{5}{12}. \end{aligned}$$

Other two functions arose from the boundary conditions [4] $f_6 = y_0 - y_2$; $f_7 = y_2 - y_6$. Choosing tolerance $1.0e^{-30}$ and number of iterations 30, the resulting graph is presented in Fig. 2.

Tab. 1 *Measured values for selected iterations DE* [5] *with* $N = 6$ AND $c = \frac{1}{6}$. Table 1

	ITERATION 1	ITERATION 3	ITERATION 5
y_0	-0.0913	-0.0970	-0.0971
y_1	-0.0709	-0.0755	-0.0755
y_2	-0.0913	-0.0970	-0.0971
y_3	-0.1227	-0.1311	-0.1311
y_4	-0.1508	-0.1617	-0.1618
y_5	-0.1568	-0.1675	-0.1675
y_6	-0.0913	-0.0970	-0.0971

For comparison, the simulation results were chosen the first, the third and the fifth iteration. The difference of values is presented in Tab. 1, it shows that the difference of the measured values after the third and fifth iteration are very small. Therefore, not all curves are visible to the naked eye in Fig. 2.

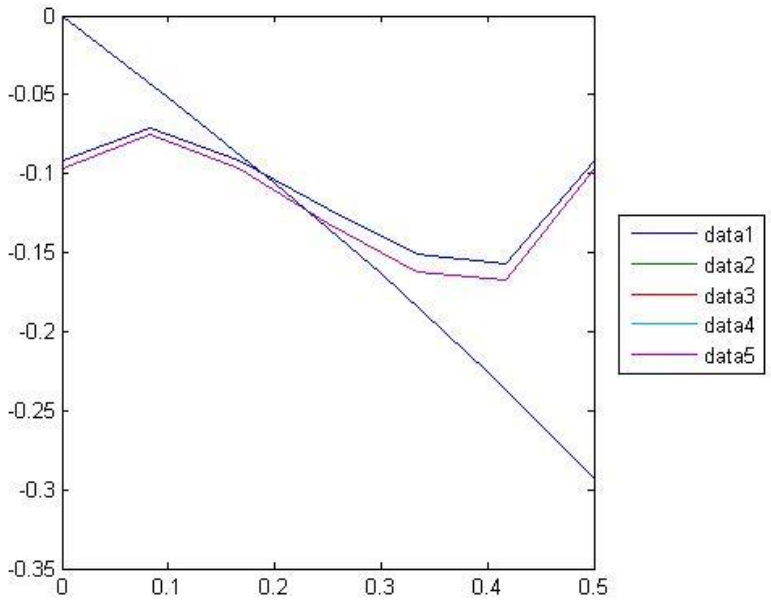

Fig. 2 *Solutions of differential equation [5] for* $N = 6$ *and function* $y = -1 + \sqrt{1 - t}$ *(black curve)*

The data for figure was worked out by using the MATLAB computer system. The source code is available from the authors upon request. Similarly, more and more points of the interval ⟨a,b⟩ and number of iterations can be chosen, and then changes in individual iterations can be observed.

DISCUSSION

In article (2), the similar solution of problem [3], [4] was accomplished. There the point c was considered in the centre of the interval ⟨a,b⟩ (Fig. 3). The transformation of the differential equation [3] to the difference equation was the same. Difference is in the function arouse from the boundary conditions [4]. If the point c is in the middle of interval ⟨a,b⟩, the function will be: $y_0 = y_{N/2}$, $y_{N/2} = y_N$..

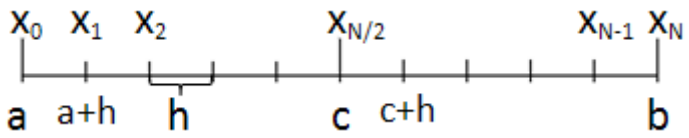

Fig. 3 *Division of the interval ⟨a, b⟩ if c is exactly in the middle of the interval*

MATLAB program was applied to the system [3], [4] with the same conditions as in the article (2). Interval $\langle 0; 1/2\rangle$ was divided into six subintervals. Functions for [5] were created, where $h = \frac{1}{12}, \varepsilon = \frac{1}{100}, k = -2$ and function $f(x, y) = y^2 + x$. Chosen was tolerance $1.0e^{-30}$ and number of iterations 30, the resulting graph is presented in Fig. 4 and the difference of values is presented in following Table 2.

MEASURED VALUES FOR SELECTED ITERATIONS *DE* [5] WITH $N = 6$ AND $c = \frac{1}{4}$ Table 2

	ITERATION 1	ITERATION 3	ITERATION 5
y_0	-0.1250	-0.1343	-0.1343
y_1	-0.0821	-0.0884	-0.0884
y_2	-0.0953	-0.1020	-0.1020
y_3	-0.1250	-0.1343	-0.1343
y_4	-0.1547	-0.1671	-0.1671
y_5	-0.1679	-0.1810	-0.1810
y_6	-0.1250	-0.1343	-0.1343

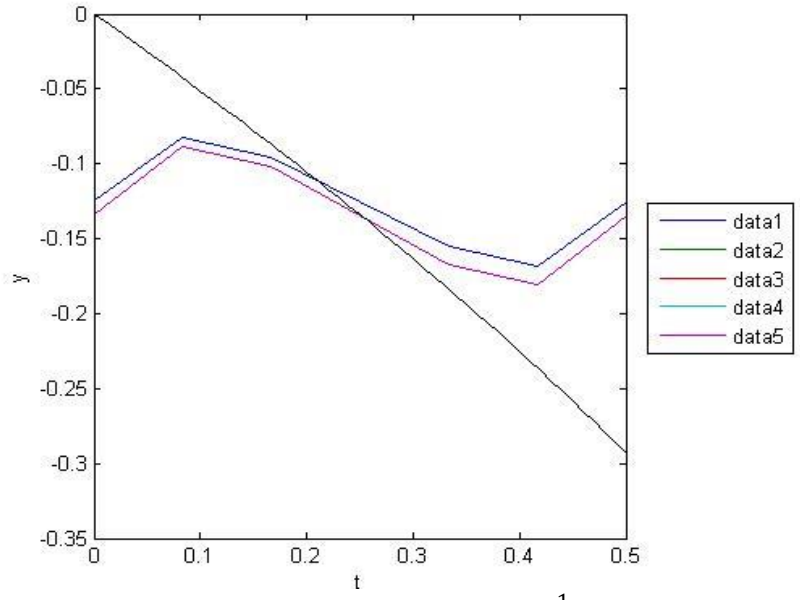

Fig. 4 *Solutions of differential equation [5] for* $N = 6$, $c = \frac{1}{4}$ *and function* $y = -1 + \sqrt{1 - t}$ *(black curve)*

As you can see, the position of point c does not have negative influence on convergence of the solutions of perturbed problems [3], [4] to the solution of reduced problem.

CONCLUSION

We proved that the solutions of perturbed problems [3], [4] converge rapidly to the solution of reduced problem for different number of points of dividing of the interval ⟨a,b⟩ with the use of Newton-Raphson method.

The proposed scheme is effective because, after five iterations, we obtain highly accurate results (1).

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