RESEARCH PAPERS
FACULTY OF MATERIALS SCIENCE AND TECHNOLOGY IN TRNAVA
SLOVAK UNIVERSITY OF TECHNOLOGY IN BRATISLAVA

2015 Volume 23, Special Number

ON THE RELATIVISTIC CORRECTION OF PARTICLES TRAJECTORY IN TANDEM TYPE ELECTROSTATIC ACCELERATOR

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Abstract

A constant potential is applied to the acceleration of the ion-beam in the tandem type electrostatic accelerator. However, not just one voltage is applied, but instead a number of applications can be made in succession by means of the tandem arrangement of high voltage tubes. This number of voltage applications, which is the number of so-called "stages" of a tandem accelerator, may be two, three, or four, depending on the chosen design. Electrostatic field with approximately constant intensity acts on ions in any stage.
In general, non-relativistic dynamics is used for the description of the ion transport in tandem accelerator. Energies of accelerated ions are too low and relativistic effects cannot be commonly observed by standard experimental technique. Estimation of possible relativistic correction of ion trajectories is therefore only a matter of calculation. In this note, we briefly present such calculation. Our aim is to show how using the relativistic dynamics modifies the particles trajectory in tandem type accelerator and what parameters determine this modification.

Key words

ion beam, relativistic correction, tandem type electrostatic accelerator, ion trajectory, relativistic dynamics

1. INTRODUCTION

The principles and techniques used in Tandem accelerator have been originated and developed over many years. Tandem accelerator facility is very simple and operative. Ions are accelerated by constant electrostatic field in this accelerator and description of their dynamics is easy mainly in non-relativistic case. In all practical applications, we only need to non-relativistic approach with respect to relatively low energies of accelerated ions. Relativistic approach could be particularly interesting in terms of knowledge of limits at the interpretation of measured results especially in cases when the ion beam transport is accompanied by light particles such as electrons (which may be relativistic).

Description of ion dynamics in both relativistic and non-relativistic cases is based on the second Newton`s law. The second Newton`s law of classical dynamics can be formally extended to relativistic form in several ways. The standard 3D relativistic dynamics equation can be simply written as (1, 2 and 3):

$$\vec{F} = \frac{d\vec{p}}{dt} \ , \qquad [1]$$

where $\vec{F}$ is the 3D force and $\vec{p} = \gamma m_0 \vec{v}$ is the 3D relativistic momentum. When the 3D force is constant, the solutions of [1] are uniformly called accelerated motion. However, equation [1] is not covariant with respect to the full Lorentz group.
In relativistic dynamics, so called 4-force (or Minkowski force) has been defined. As a 4D covariant extension of [1], we have:

$$\vec{F} = \frac{d\vec{p}}{d\tau} \ , \qquad [2]$$

where $\vec{F}$ is 4-force, $\vec{p}$ is the 4-momentum, and τ is proper time in this case (see (3)). In contrast to ordinary vector force, the 4-force components transform in the same way as any 4-vector. We do not need to know neither velocity nor 4-force components in one frame to find the 4-force components in another frame. However, until one knows how to calculate the applied force (for example in terms of electric field) definition of 4-force is practically entirely the same as in 3D form. Unfortunately, equation [2] is also covariant only with respect to the "little" Lorentz group and not with respect to the full Lorentz group. Moreover, when *F* is a constant, equation [2] has no solution. This follows from the fact that the 4-velocity and the 4-acceleration are perpendicular (4, 5). For that reason, the correction of the definition of "uniformly acceleration" was needed in relativistic dynamics. This was necessary to find a fully Lorentz covariant relativistic dynamics equation for a better concept of “uniform acceleration.”
Physical definition of "uniformly accelerated motion" is the motion whose acceleration is constant in the co-moving frame. This definition is found widely in the literature, as early as (6) and (7), again in (8), and recently in (9) and (10). 4D Lorentz covariant Relativistic Dynamics Equation allowing to describe this definition mathematically in correct form is:

$$c\frac{du^{\mu}}{d\tau} = A^{\mu}_{\nu} u^{\nu} \ , \qquad [3]$$

where A^{μ}_{ν} is antisymmetric rank 2 tensor, *u* is 4-velocity. We refer to *A* as the acceleration tensor associated with the given motion. By taking the tensor *A* to have constant components, we obtain a covariant definition of "uniformly accelerated" motion.
To calculate relativistic correction to the ion beam trajectory, it is convenient to start from the simplest formula [1] despite the difficulties with covariance described above. We applied 3D formula [1] to estimate the relativistic correction of ion-beam trajectory in tandem electrostatic accelerator. Despite the fact that relativistic effect has no practical significance, only relativistic ion-beam dynamics should be applied to obtain indeed the correct results.

2. NON-RELATIVISTIC CASE

The non-relativistic equation of motion for the ion with mass *m* and electric charge *q* moving along the *x*-axis in electrostatic field with constant intensity *E* oriented in the *x*-direction can be found by second Newton`s law [1]:

$$m\frac{dv}{dt} = qE \ . \qquad [4]$$

The solution to this equation determines the ion velocity as a function of time *v(t)*. Next, it is easy to find position of the ion as a function of time by integrating:

$$x(t)=\frac{qE}{2m}t^2+v_0t\,, \qquad [5]$$

where v_0 is initial ion velocity. For the ion velocity in the radial direction (in direction of y axis), it can be written:

$$v_y=\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}=y'v \qquad [6]$$

and the initial ion velocity can be determined considering the ion kinetic energy on entry into the accelerator where the potential of electrostatic field is φ_1:

$$v_0=\sqrt{\frac{2q\varphi_1}{m}}\,. \qquad [7]$$

No force acts on the ion in the radial direction. For that reason, it implies:

$$y(t)=v_yt=(y')_{t=0}\,t\sqrt{\frac{2q\varphi_1}{m}}\,. \qquad [8]$$

Non-relativistic equation for ion trajectory can be determined by elimination of time t from equations (5) and (8), i.e.:

$$t=\frac{y}{(y')_{t=0}}\sqrt{\frac{m}{2q\varphi_1}} \quad \text{and next:} \quad x=\frac{qE}{2m}\left(\frac{y}{(y')_{t=0}}\sqrt{\frac{m}{2q\varphi_1}}\right)^2+\sqrt{\frac{2q\varphi_1}{m}}\left(\frac{y}{(y')_{t=0}}\sqrt{\frac{m}{2q\varphi_1}}\right)\,. \qquad [9]$$

Formula (9) can be rewritten to the form:

$$y=2\left(\frac{\varphi_1}{U}\right)L\left\{\sqrt{1+\frac{1}{L}\left(\frac{U}{\varphi_1}\right)x}-1\right\}(y')_{t=0}\,. \qquad [10]$$

where U is voltage applied on the accelerator and L is accelerator length. Since this is a homogeneous electrostatic field in tandem accelerator, we used:

$$E=\frac{U}{L}\,. \qquad [11]$$

Ion beam trajectory equation in non-relativistic case can be determined by using equation [10]:

3. RELATIVISTIC CASE

In relativistic case, the relativistic momentum must be considered in (1). Equation of motion takes the form:

$$\frac{d}{dt}\left(\frac{m_0\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)=q\vec{E}\,, \qquad [12]$$

where m_0 is rest mass of ion. After calculating derivations on left side of equation [12], we get:

$$\overline{\overline{m}}.\vec{a}=q\vec{E}\,, \qquad [13]$$

where tensor $\overline{\overline{m}}$ can be written as:

$$\overline{\overline{m}}=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\overline{\overline{I}}+\frac{m_0\vec{v}\vec{v}}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\,. \qquad [14]$$

In case of longitudinal transport (i.e. if vectors $\vec{a}$ is parallel with $\vec{v}$), equation of motion [12] goes into the next separable differential equation:

$$\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\frac{dv}{dt}=F\ . \qquad [15]$$

Equation (15) can be solved by simple integration and we get a formula for the ion velocity calculation in the form:

$$v(t)=\frac{c\left(qEt+m_0K\right)}{\sqrt{\left(qEt+m_0K\right)^2+m_0^2c^2}}\ , \qquad [16]$$

where K is integration constant resulting from the initial conditions. We know that at the beginning (i.e. at the t = 0), the ion velocity is v_0. If we apply it in (16) we get:

$$K=\frac{v_0}{\sqrt{1-\frac{v_0^2}{c^2}}}\ . \qquad [17]$$

Using basic kinematic principles we can calculate the ion position by time integration of formula (16):

$$x(t)=\int v(t)dt=\frac{c}{qE}\sqrt{\left(qEt+m_0K\right)^2+m_0^2c^2}+K'\ , \qquad [18]$$

where K' is integration constant. Whereas $x(t=0)=0$, this constant is:

$$K'=-\frac{m_0c}{qE}\sqrt{K^2+c^2}\ . \qquad [19]$$

After considering relativistic kinetic energy of ion on entry into the accelerator where the potential of electrostatic field is φ_1, the initial ion velocity can be found:

$$v_0=c\sqrt{1-\left(\frac{m_0c^2}{m_0c^2+q\varphi_1}\right)^2}\ . \qquad [20]$$

If we consider [17], [19] and [20] in [18], we get the ion position in the form:

$$x(t)=\frac{c^2L}{U}\left\{\sqrt{\left[\frac{U}{cL}t+\sqrt{\left(\frac{m_0}{q}+\frac{\varphi_1}{c^2}\right)^2-\frac{m_0^2}{q^2}}\right]^2+\frac{m_0^2}{q^2}}-\left(\frac{m_0}{q}+\frac{\varphi_1}{c^2}\right)\right\}\ . \qquad [21]$$

Next, we can apply the standard procedure as in the non-relativistic case. Equation [8] in relativistic case enters into the form:

$$y(t)=\left(y'\right)_{t=0}tc\sqrt{1-\left(\frac{m_0c^2}{m_0c^2+q\varphi_1}\right)^2}\ . \qquad [22]$$

Relativistic equation for ion trajectory can be found by elimination of time t from equations [21] and [22], i.e.:

$$y=\frac{\sqrt{\left(\frac{m_0}{q}+\frac{\varphi_1}{c^2}\right)^2-\frac{m_0^2}{q^2}}}{U\left(\frac{m_0}{q}+\frac{\varphi_1}{c^2}\right)}\left\{\sqrt{\left\{\left(\frac{U}{c^2L}\right)x+\left(\frac{m_0}{q}+\frac{\varphi_1}{c^2}\right)\right\}^2-\frac{m_0^2}{q^2}}-\sqrt{\left(\frac{m_0}{q}+\frac{\varphi_1}{c^2}\right)^2-\frac{m_0^2}{q^2}}\right\}c^2L\left(y'\right)_{t=0}\ . \qquad [23]$$

It is possible to find ion beam trajectory in relativistic case by means of equation [23]. This result shows that in the relativistic case ion trajectory depends on the mass and electric charge of the ions. Therefore, trajectories are not the same for all type of ions. Comparison of trajectories in relativistic and non-relativistic case can be carried out by formulas [10] and [23]. Results are shown in the following graphs.

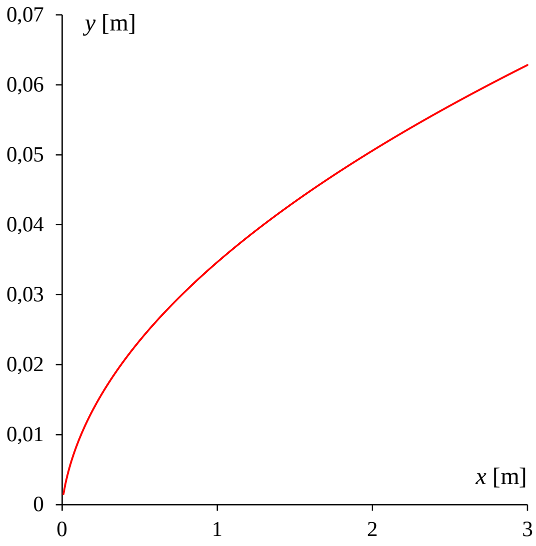

Fig. 1 *Radial deflection as a function of position for ions* Li^{1+}, Br^{1+}, O^{1+} *and* He^{1+} *in tandem accelerator*

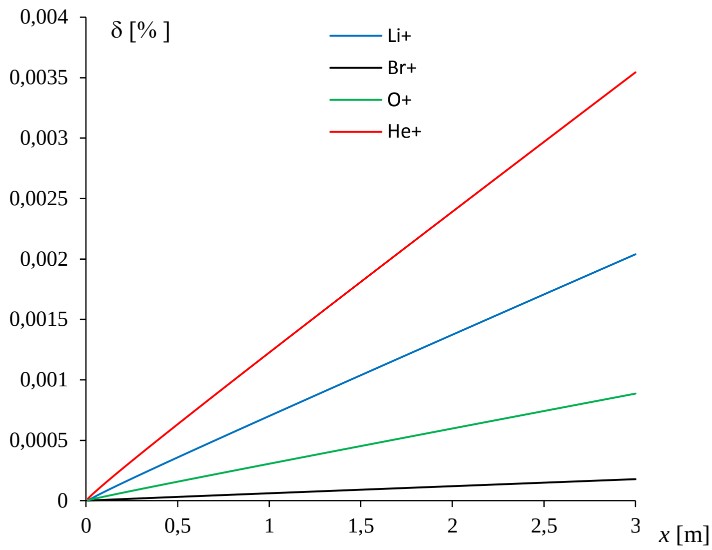

Fig. 2 *Relativistic correction of trajectory for ions* Li^{1+}, Br^{1+}, O^{1+} *and* He^{1+}

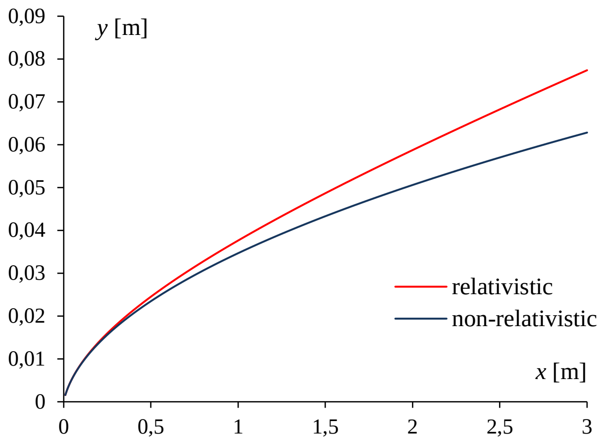

Fig. 3 *Radial deflection as a function of position for electron in tandem accelerator*

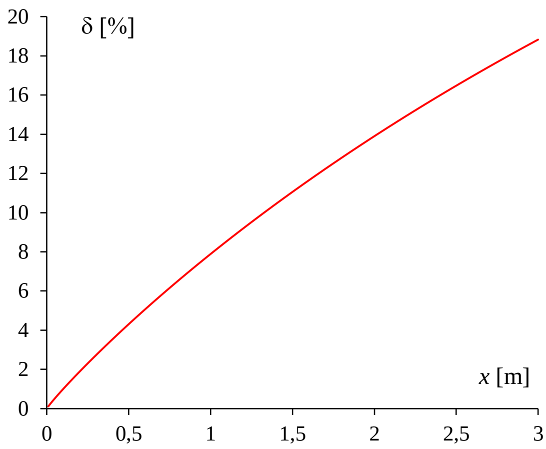

Fig. 4 *Relativistic correcton of trajectory for electron*

Radial deflections of trajectories calculated from formulas [10] and [23] for ions Li^{1+}, Br^{1+}, O^{1+} and He^{1+} are shown in Fig.1. Relativistic correction of these trajectories shown in Fig.2 were determined by formula:

$$\delta = \left(\frac{y_{relativistic} - y_{nonrelativistic}}{y_{relativistic}} \right) .100\% \ .$$

Next parameters were used for calculation: initial deflection angle: 10° from which we get $(y')_{t=0} = 0.174532925$, voltage $U = 500$ kV, potential $\varphi_1 = 2$ kV, speed of light $c = 3.10^8$ m.s^{-1} accelerator length $L = 3$ m, charge of electron (ions) $e = 1.602.10^{-19}$ C. Rest mass of each particle is given in Table 1.

Table 1

Particle	Rest mass [kg]
Li^{1+}	$1,153.10^{-26}$
Br^{1+}	$13,268.10^{-26}$
O^{1+}	$2,657.10^{-26}$
He^{1+}	$6,65.10^{-27}$
Electron	$9,11.10^{-31}$

Radial deflections of ions Li^{1+}, Br^{1+}, O^{1+} and He^{1+} are almost identical and graphs for each ion in the Fig.1 overlap. Relative corrections in Fig. 2 are very small and differences cannot be distinguished. Relativistic and non-relativistic trajectory for each ion is almost same in this case.

The situation is different for electron. As can be seen from Fig. 3, relativistic trajectory for electron differs significantly from non-relativistic and relative corrections reach up to nearly 20 % (see Fig. 4).

4. CONCLUSION

3D relativistic dynamic equation was used to obtain ion-beam trajectory in the tandem type electrostatic accelerator. We turn our attention on evaluation of relativistic effect in the trajectories at optional accelerating voltages *U*. Explicit solution was obtained in the case of constant electrostatic force acting on the accelerated particle. Results showed that relativistic corrections play a role only in accelerating very light particles such as electron.

In conclusion, we should like to call attention to the fact that, in the contrast with non-relativistic case, relativistic ion-beam trajectories are sensitive to ion-beam characteristics (mass and charge of ions). It is evident from equation [23].

Acknowledgement

This research was funded by the ERDF - Research and Development Operational Programme under the project of "University Scientific Park Campus MTF STU - CAMBO" ITMS: 26220220179.

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