ON STABILITY OF CERTAIN CLASS OF 3-RD ORDER NONLINEAR CONTROL SYSTEMS

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Abstract

The paper deals with a certain class of nonautonomous ordinary third-order nonlinear differential equations \( L_3 y = f(t, L_0 y, L_1 y, L_2 y) \) with quasi-derivatives. A criterion of asymptotic stability in Liapunov sense as well as a criterion of instability in Liapunov sense is derived. The results are illustrated by two examples.

Key words

nonautonomous nonlinear differential equation; 3-rd order; quasi-derivative; stability in Liapunov sense; asymptotic stability in Liapunov sense, instability in Liapunov sense

Introduction

In the frame of Control Theory, Palumbíny in (5) derived stability criteria of certain class of third-order nonlinear differential equations with so called quasi-derivatives (see the text below). Then we stated such results for autonomous nonlinear 3-rd equations (see (7)). The main aim of the presented article is to establish similar criteria for another class of third-order nonlinear differential equations with quasi-derivatives. Our paper deals with a criterion of asymptotic stability as well as instability, both in Liapunov sense, of a null solution 0 of the nonautonomous nonlinear third-order differential equations with the quasi-derivatives

\[
L_3 y = f(t, L_0 y, L_1 y, L_2 y),
\]

where (the prime means a derivative owing to the variable \( t \))
\[ L_0 y(t) = y(t), \]
\[ L_1 y(t) = p_1(t)(L_0 y(t))', \]
\[ L_2 y(t) = p_2(t)(L_1 y(t))', \]
\[ L_3 y(t) = (L_2 y(t))'. \]

\( p_i(t), i = 1, 2 \) are real-valued continuous functions defined on an open interval \((b, \infty)\), \( f(t, y_1, y_2, y_3) \) is real-valued and continuous up to all 2nd order partial derivatives of the function \( f \) owing to \( y_k, k = 1, 2, 3 \). A symbol \( E \) denotes the set of all real numbers. The symbol \( \mathbf{o} \) means the vector \((0,0,0)\). The symbol 0 is, according to a situation, a real number null or a real null function. The terms \( L_k y(t), k = 0,1,2,3 \) are so called \( k \)-th quasi-derivatives of a function \( y(t) \).

We note that we shall use a matrix norm of the form \( \|\{a_{ij}\}_{i,j}\| = \sum_{i,j} |a_{ij}| \). A set \([-1,1]\) is a closed interval with bounds \(-1, 1\). An important contribution of the article consists in the fact that there are more control parameters in [L]. These parameters are the functions \( p_i(t), i = 1,2 \) which enable an user a better control of considered processes described by the equation [L].

**Remark 1.** The equation [L] can be expressed more detailed as

\[ [M] \quad \left( p_2(t)(p_1(t)y')' \right)' = f\left(t, y, p_1(t)y', p_2(t)(p_1(t)y')' \right). \]

Now we determine notions concerning the stability resp. instability in Liapunov sense. Let us consider a general differential system of the first order

\[ [S] \]
\[ y'_1 = f_1(t, y_1, y_2, y_3) \]
\[ y'_2 = f_2(t, y_1, y_2, y_3) \]
\[ y'_3 = f_3(t, y_1, y_2, y_3) \]

**Assumption.** Let the system \([S]\) be expressed in a matrix form \( \mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \). Throughout the paper we shall assume an existence of a number \( b \) (real or \(-\infty\)) and an area \( H \subseteq E^3 \), \( \mathbf{o} \in H \) such that the function \( \mathbf{f} \) is continuous on a set \( G = (b, \infty) \times H \) and for every point \((\tau, \mathbf{k}) \in G \) the following Cauchy problem

\[ [1] \]
\[ \mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(\tau) = \mathbf{k}, \]

admits the only solution. We also assume \( \mathbf{f}(t, \mathbf{0}) = \mathbf{0} \) for all \( t > b \), i.e. the Cauchy problem \([1]\) admits for \( \mathbf{k} = \mathbf{0} \) the trivial solution defined by a formula \( \mathbf{0}(t) = \mathbf{0} \) for all \( t > b \).

**Definition 1.** We say that the trivial solution \( \mathbf{0} \) of the system \([S]\) is stable in Liapunov sense, if for every \( \tau > b \) and every \( \varepsilon > 0 \) there exists \( \delta = \delta(\tau, \varepsilon) > 0 \) such that for every initial values \( \mathbf{k} \in H, ||\mathbf{k}|| < \delta \) and for all \( t \geq \tau \) it holds that the solution \( \mathbf{u}(t, \tau, \mathbf{k}) \) of the Cauchy problem \([1]\) fulfills the following inequality
\[ \| u(t, \tau, k) \| < \varepsilon. \]

Otherwise, the trivial solution \( o \) is instable in Liapunov sense.

**Definition 2.** We say that the trivial solution \( o \) of the system \([S]\) is asymptotically stable in Liapunov sense, if the trivial solution \( o \) is stable in Liapunov sense and there exists a real number \( \Delta > 0 \) such that for all \( k \in H, \| k \| < \Delta \) and every \( \tau > b \) it holds that

\[ \lim_{t \to \infty} \| u(t, \tau, k) \| = 0. \]

**Definition 3.** A special type \([T]\) of the system \([S]\) of the form

\[
\begin{align*}
y_1' &= y_2 p_1(t) \\
y_2' &= y_1 p_2(t) \\
y_3' &= f(t, y_1, y_2, y_3)
\end{align*}
\]

is called a competent system to the equation \([L]\).

**Remark 2.** We recall an important property of the system \([T]\) which consists in a fact that a function \( u(t) \) is a solution of \([L]\) if and only if a vector \((u(t), L_1 u(t), L_2 u(t))\) is a solution of \([T]\).

**Definition 4.** Let \([L]\) be such an equation that the function \( 0 \) is a solution of \([L]\) on \((b, \infty)\). Then, according to Remark 2, the vector \((0,0,0)\) is a solution of \([T]\). We say \( 0 \) is a stable solution of \([L]\) in Liapunov sense, if \((0,0,0)\) is a stable solution of \([T]\) in Liapunov sense. Otherwise, \( 0 \) is an instable solution of \([L]\) in Liapunov sense.

**Definition 5.** Let \([L]\) be such an equation that \( 0 \) is a solution of \([L]\) on \((b, \infty)\). Then, according to Remark 2, the vector \((0,0,0)\) is a solution of \([T]\). We say \( 0 \) is an asymptotically stable solution of \([L]\) in Liapunov sense, if \((0,0,0)\) is an asymptotically stable solution of \([T]\) in Liapunov sense.

The main aim of the paper is to establish the criteria, which assure the asymptotic stability as well as instability of the null solution \( 0 \) of the equation \([L]\). If we put \( p_k(t) = 1 \) on \((b, \infty), \ k = 1, 2\) in \([L]\), we obtain a differential equation with classic derivatives. We note that the functions \( p_k(t), \ k = 1, 2\) are not, in general, assumed to be differentiable. From this it follows that we cannot use on \([L]\) stability criteria derived for nonlinear differential equations with classic derivatives.

### Auxiliary assertions

Now we introduce some auxiliary assertions which are significant for our considerations. The first of them is the special case of the wellknown Hurwitz criterion when \( n = 3\) (see (2), Chapter 9 or (3)):
Theorem 1. Let us consider a polynomial

\[ b_3 s^3 + b_2 s^2 + b_1 s + b_0, \]

where \( b_i, i = 0, 1, 2, 3 \) are real numbers such that \( b_0 > 0, b_1 \neq 0 \). Then all zeros of the polynomial [3] admit negative real parts if and only if it holds that

\[ b_1 > 0, \]
\[ b_1b_2 - b_0b_3 > 0, \]
\[ b_1b_2b_3 - b_0b_2^2 > 0. \]

The second assertion deals with asymptotic criteria of stability in Liapunov sense for systems of differential equations of the first order (see (1), Chapter 13 or (4)):

Theorem 2. Let us consider a system of differential equations of the first order expressed in the following matrix form

\[ x' = Ax + B(t)x + g(t, x), \quad g(t, o) = o, \]

where \( A \) is a real constant square matrix, \( B(t) \) is a real square matrix depending on \( t \) only, such that

\[ \lim_{t \to \infty} B(t) = 0, \]

where \( 0 \) is a null matrix and \( g \) a real vector function continuous on an area \( (b, \infty) \times H \), where \( b \in E \), satisfying a condition

\[ \lim_{|x| \to 0} \frac{\|g(t, x)\|}{\|x\|} = 0 \text{ uniformly for all } t \geq b. \]

Then:

[i] If all eigenvalues of \( A \) have negative real parts, then the trivial solution \( o \) of [7] is asymptotically stable in Liapunov sense.

[ii] If at least one of eigenvalues of \( A \) has a positive real part, then the trivial solution \( o \) of [7] is instable in Liapunov sense.

Theorem 3. Let \( \lim_{|x| \to 0} \frac{\|g_1(t, x)\|}{\|x\|} = 0 \) uniformly for all \( t \geq b \) as well as let \( \lim_{|x| \to 0} \frac{\|g_2(t, x)\|}{\|x\|} = 0 \) uniformly for all \( t \geq b \). Then \( \lim_{|x| \to 0} \left(\frac{\|g_1(t, x)\|}{\|x\|} + \frac{\|g_2(t, x)\|}{\|x\|}\right) = 0 \) uniformly for all \( t \geq b \).

Proof. The proof immediately follows from the fact, that a norm of sum of two functions is less than or equal to sum of norms of these functions for any \( t \geq b \).

Results

Now we shall prove the first main result of the paper – the criterion of asymptotic stability of the null solution \( o \) in Liapunov sense of the differential equation [L]:

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Theorem 4. Let us consider the differential equation \([L]\) such that

\[ \lim_{t \to \infty} p_i(t) = a_i > 0, \quad i = 1, 2. \]

Let \( w = (w_1, w_2, w_3) \). If it holds that

\[ \lim_{t \to \infty} f(t, w_1, w_2, w_3) = f(\infty, w_1, w_2, w_3), \]

\[ \frac{\partial f}{\partial w_1}(\infty, o) < 0, \quad \frac{\partial f}{\partial w_2}(\infty, o) < 0, \quad a_1 \frac{\partial f}{\partial w_2}(\infty, o) + \frac{\partial f}{\partial w_1}(\infty, o) > 0, \]

\[ \lim_{||w|| \to 0} \left| f(t, w_1, w_2, w_3) - f(\infty, w_1, w_2, w_3) \right| = 0 \quad \text{uniformly for} \quad t \in (b, \infty), \]

then the null solution 0 of \([L]\) on \((b, \infty)\) is asymptotically stable in Liapunov sense.

Proof. The equation \([L]\) can be expressed in the form

\[ L_3 y = f(\infty, L_0 y, L_1 y, L_2 y) + [f(t, L_0 y, L_1 y, L_2 y) - f(\infty, L_0 y, L_1 y, L_2 y)]. \]

The null solution 0 is, according to Definition 5, asymptotically stable in Liapunov sense, if the solution \( o \) of the system \([T]\) is asymptotically stable in Liapunov sense, where \([T]\) is expressed in the form \([U]\), where

\[ [U] \quad w' = Aw + B(t)w + g(t, w), \quad g(t, o) = o. \]

and

\[ w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \quad w' = \begin{bmatrix} w'_1 \\ w'_2 \\ w'_3 \end{bmatrix}, \quad o = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0, & \frac{1}{a_1}, & 0 \\ 0, & 0, & \frac{1}{a_2} \\ \frac{\partial f}{\partial w_1}(\infty, o), & \frac{\partial f}{\partial w_2}(\infty, o), & \frac{\partial f}{\partial w_3}(\infty, o) \end{bmatrix}, \]

\[ B(t) = \begin{bmatrix} 0, & \frac{1}{p_1(t)} - \frac{1}{a_1}, & 0 \\ 0, & 0, & \frac{1}{p_2(t)} - \frac{1}{a_2} \\ 0, & 0, & 0 \end{bmatrix}, \quad g(t, w) = g_1(t, w) + g_2(t, w), \]

\[ g_1(t, w) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \left( w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2} + w_3 \frac{\partial}{\partial w_3} \right)^2 f(\infty, \Theta w) \end{bmatrix}, \quad g_2(t, w) = \begin{bmatrix} 0 \\ 0 \\ f(t, w) - f(\infty, w) \end{bmatrix}, \]

where we used on the function \( g_i(t, w) \) which does not depend on the variable \( t \) the Taylor’s theorem with the remainder in the Lagrange’s form as \( k = 2, 0 < \Theta < 1 \). Then
0 \leq \frac{\|g_1(t, w)\|}{\|w\|} = \frac{|0| + |0| + \frac{1}{2} \left( w_1 \frac{\partial f}{\partial w_1} + w_2 \frac{\partial f}{\partial w_2} + w_3 \frac{\partial f}{\partial w_3} \right)^2 f(\infty, \Theta w)}{\|w\|} \leq \frac{1}{|w_1| + |w_2| + |w_3|} \left( |\frac{1}{2} \frac{\partial^2 f}{\partial^2 w_1^2} (\infty, \Theta w) w_1|^2 + \frac{1}{2} \frac{\partial^2 f}{\partial^2 w_2^2} (\infty, \Theta w) w_2|^2 + \frac{1}{2} \frac{\partial^2 f}{\partial^2 w_3^2} (\infty, \Theta w) w_3|^2 \right) + \frac{\partial^2 f}{\partial w_i \partial w_j} (\infty, \Theta w) w_i w_j \right) + \frac{\partial^2 f}{\partial w_i \partial w_j} (\infty, \Theta w) w_i w_j + \frac{\partial^2 f}{\partial w_i \partial w_j} (\infty, \Theta w) w_i w_j \right)

Without loss of generality, we can assume that \((w_1, w_2, w_3) \in [-1,1]^3\). From this and from the continuity of all 2-nd order partial derivatives of the function \(f\) it follows that there exists a positive real constant \(K\) such that

\[
0 \leq \frac{\|g_1(t, w)\|}{\|w\|} \leq \frac{K}{|w_1| + |w_2| + |w_3|} \left( |w_1|^2 + |w_2|^2 + |w_3|^2 + |w_1| w_2 + |w_1| w_3 + |w_2| w_3 + |w_3| w_1 \right) = K \left( |w_1| + |w_2| + |w_3| \right) \leq K \|w\|.
\]

The squeeze theorem from Limit Theory yields that \(\frac{\|g_1(t, w)\|}{\|w\|}\) converges to zero as \(\|w\| \to 0\).

This convergence is uniform because the term \(K \left( |w_1| + |w_2| + |w_3| \right)\) does not explicitly depend on the variable \(t\). From [b] as well as [d] we obtain \(\frac{\|g_2(t, w)\|}{\|w\|}\) uniformly converges to zero as \(\|w\| \to 0\). Then Theorem 3 yields that [9] hold. We can easily observe a validity of the conditions [7], [8] in Theorem 2. The validity of condition [a], [c] assure that the conditions [4], [5], [6] hold in Theorem 1, where characteristic polynomial of the matrix \(A\) is

\[
s^3 - \frac{\partial f}{\partial w_3} (\infty, \Theta) s^2 - \frac{1}{a_2} \frac{\partial f}{\partial w_2} (\infty, \Theta) s - \frac{1}{a_1 a_2} \frac{\partial f}{\partial w_1} (\infty, \Theta).
\]

Then Theorem 1 yields that all the eigenvalues of \(A\) have the negative real parts. Consequently, Theorem 2, the part [i] as well as Definition 5 yield the required stability of the null solution \(0\) of [L].

**Example 1.** Let us consider the differential equation [L], where \(p_1(t) = 2 + \frac{1}{t},\ p_2(t) = 3 - \frac{1}{t}\) and

\[
L_3 y = f(t, L_0 y, L_1 y, L_2 y) = \frac{y^4}{1 + (L_2 y)^2} \left( 1 + \frac{2}{t} \right) - y - 2L_1 y - 3L_2 y.
\]

Then
\[ f(\infty, w_1, w_2, w_3) = \lim_{t \to \infty} f(t, w_1, w_2, w_3) = \frac{y^4}{1 + (L_2 y)^2} - y - 2L_1 y - 3L_2 y. \]

It is obvious that Assumption, mentioned after Remark 1, hold for \( b = 2 \). An easy computing yields that the nonlinear differential equation [L] admits the null solution 0 as well as \( a_1 = 2, a_2 = 3, \frac{\partial f}{\partial w_1}(\infty, o) = -1, \frac{\partial f}{\partial w_2}(\infty, o) = -2, \frac{\partial f}{\partial w_3}(\infty, o) = -3 \). From this immediately follows the validity of [a], [b] and [c] in Theorem 4. Then

\[
0 \leq \left\| \frac{f(t, w_1, w_2, w_3) - f(\infty, w_1, w_2, w_3)}{\| w \|} \right\| \leq \frac{\left| w_1^4 \cdot \frac{2}{1 + w_2^2} \right|}{\| w \|} \leq \frac{\left| w_1^4 \cdot \max_{i=2} \left( \frac{2}{i} \right) \right|}{\| w \|} = \frac{w_1^4 \cdot 1}{\| w \|} \leq \frac{|w_1| w_3^3}{\| w \|} \leq |w_3|^3.
\]

From this immediately follows \( \lim_{\| w \| \to 0} \left| f(t, w_1, w_2, w_3) - f(\infty, w_1, w_2, w_3) \right| = 0 \) uniformly for \( t \in (b, \infty) \) because of the expression \( |w_3|^3 \) does not depend on the variable \( t \). Consequently, the condition [d] in Theorem 4 holds. Then the last mentioned theorem yields required stability of the null solution 0 of the equation [L].

Now we shall prove the second main result of the paper – the criterion of instability of the null solution 0 in Liapunov sense of the differential equation [L]:

**Theorem 5.** Let us consider the differential equation [L] such that

[a] \( \lim_{t \to \infty} p_i(t) = a_i > 0, \ i = 1, 2. \)

If it hold that

[b] \( \lim_{t \to \infty} f(t, w_1, w_2, w_3) = f(\infty, w_1, w_2, w_3), \)

[c'] at least one real part of zeros of \( s^3 - \frac{\partial f}{\partial w_3}(\infty, o) s^2 - \frac{1}{a_2} \frac{\partial f}{\partial w_2}(\infty, o) s - \frac{1}{a_1a_2} \frac{\partial f}{\partial w_1}(\infty, o) \)

is positive,

[d] \( \lim_{\| w \| \to 0} \left| f(t, w_1, w_2, w_3) - f(\infty, w_1, w_2, w_3) \right| = 0 \) uniformly for \( t \in (b, \infty), \)

then the null solution 0 of the equation [L] is instable in Liapunov sense.

**Proof.** The null solution 0 of [L] is, according to Definition 4, instable in Liapunov sense, if the solution (0,0,0) of the system [U] is instable in Liapunov sense. By the same way as in the proof of Theorem 4, it can be proved the validity of [9]. We can easily observe the validity of the conditions [7], [8] in Theorem 2. Then the last mentioned Theorem, the part [ii] yields the required instability of the null solution 0 of [L].
Example 2. Let us consider the equation \([L]\), where \(p_1(t) = 2 + \frac{1}{t}\), \(p_2(t) = 3 - \frac{1}{t}\) and

\[
L_y = \frac{y^6}{4 + (L_2 y)^2} \left(1 + \frac{2}{t}\right) + (y - 2L_y y - 3L_2 y).
\]

Then

\[
f(\infty, w_1, w_2, w_3) = \lim_{t \to \infty} f(t, w_1, w_2, w_3) = \frac{y^6}{1 + (L_2 y)^2} - y - 2L_y y - 3L_2 y.
\]

It is obvious that Assumption, mentioned after Remark 1, hold for \(b = 2\). An easy computing yields that the function \(0\) is the null solution of the differential equation \([L]\) as well as

\[
a_1 = 2, a_2 = 3, \frac{\partial f}{\partial w_1}(\infty, o) = 1, \frac{\partial f}{\partial w_2}(\infty, o) = -2, \frac{\partial f}{\partial w_3}(\infty, o) = -3.
\]

From this immediately follows the validity of \([a]\) and \([d]\). By the same way as in Example 1 it can be proved the property \([c']\). Then the characteristic polynomial of \(A\) is

\[
h(s) = s^3 + 3s^2 + \frac{2}{3}s - \frac{1}{6}.
\]

There are two possibilities only: 1) \(h(s)\) admits three real zeros. Then their product is equal to \(1/6\). It means, that all these zeros differ null. If all these zeros were negative, then their product would be negative, which is a contradiction. 2) \(h(s)\) admits one real zero \(a\) and two complex zeros \(b \pm ci\). Then their product \(a(b^2 + c^2)\) equals to \(1/6\) again. If \(a \leq 0\), then this product would be nonpositive. Thus \(a > 0\). Then, owing to Theorem 5, the function \(0\) is an instable solution of \([L]\) in Lyapunov sense.

**Conclusion**

The foregoing results can be used for ordinary nonlinear differential equations with quasi-derivatives. Especially, when the functions \(p_1(t), p_2(t)\) are not differentiable on a considered interval, where classical stability criteria cannot be used. The differential equations in applications where the quasi-derivatives have been occurred are, for example, the differential equations describing a stationary distribution of temperature in a wall of a circle tube as well as the differential equations of an equilibrium state of a straight mass bar. For more details see (6).

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