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SINGULARLY PERTURBED LINEAR NEUMANN PROBLEM WITH THE CHARACTERISTIC ROOTS ON THE IMAGINARY AXIS

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Abstract

We investigate the problem of existence and asymptotic behavior of solutions for the singularly perturbed linear Neumann problem

$$\varepsilon y'' + ky = f(t), \quad k > 0, \qquad 0 < \varepsilon << 1, \quad t \in \langle a, b \rangle$$
$$y'(a) = 0, \qquad y'(b) = 0.$$

Our approach relies on the analysis of integral equation equivalent to the problem above.

Key words

singularly perturbed ODE, Neumann problem, boundary condition, characteristic roots

Introduction

In this paper, we will study the singularly perturbed linear problem

$$\varepsilon y'' + ky = f(t), \quad k > 0, \qquad 0 < \varepsilon << 1, \quad f \in C^3(\langle a, b \rangle) \tag{1.1}$$

with Neumann boundary condition

$$y'(a) = 0, \qquad y'(b) = 0 . \tag{1.2}$$

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We can view this equation as a mathematical model of the dynamical systems with high-speed feedback. The situation considered here is complicated by the fact that a characteristic equation of this differential equation has roots on the imaginary axis i.e. the system be not hyperbolic. For hyperbolic ones the dynamics close critical manifold is well-known (see e.g. [1], [3-10]), but for the non-hyperbolic systems the problem of existence and asymptotic behaviour is open in general and leads to the substantial technical difficulties in nonlinear case [2]. The considerations below may be instructive for these ones.

We prove, that there exist infinitely many sequences $\{\varepsilon_n\}_{n=0}^{\infty}$, $\varepsilon_n \to 0^+$ such that $y_{\varepsilon_n}(t)$ converges uniformly to $u(t)$ on $\langle a,b\rangle$ where y_ε is a solution of problem (1.1), (1.2) and *u* is a solution of reduced problem (when we put $\varepsilon = 0$ in (1.1)) $ku = f(t)$ i.e. $u(t) = \dfrac{f(t)}{k}$.

We will consider for the parameter ε the set J_n only,

$$J_n = \left\langle k\left(\frac{b-a}{(n+1)\pi - \lambda}\right)^2, \; k\left(\frac{b-a}{n\pi + \lambda}\right)^2 \right\rangle \quad n = 0,1,2,\dots,$$

where $\lambda > 0$ be arbitrarily small but fixed constant which guarantees the existence and uniqueness of the solutions of (1.1), (1.2).

Example. Consider the linear problem

$$\varepsilon y'' + ky = e^t, \quad k > 0, \qquad 0 < \varepsilon << 1, \quad t \in \langle a,b\rangle$$
$$y'(a) = 0, \qquad y'(b) = 0.$$

and its solution

$$y_\varepsilon(t) = \frac{-e^a \cos\left[\sqrt{\frac{k}{\varepsilon}}(b-t)\right] + e^b \cos\left[\sqrt{\frac{k}{\varepsilon}}(t-a)\right]}{\sqrt{\frac{k}{\varepsilon}}(k+\varepsilon)\sin\left[\sqrt{\frac{k}{\varepsilon}}(b-a)\right]} + \frac{e^t}{k+\varepsilon}.$$

Hence, for every sequence $\{\varepsilon_n\}_{n=0}^{\infty}, \varepsilon_n \in J_n$ the solution of considered problem

$$y_{\varepsilon_n}(t) = \frac{e^t}{k+\varepsilon_n} + O\left(\sqrt{\varepsilon_n}\right)$$

converges uniformly for $n \to \infty$ to the solution $u(t) = \dfrac{e^t}{k}$ of the reduced problem on $\langle a,b\rangle$.

The main result of this article is the following one.

Main result

Theorem. *For all* $f \in C^3(\langle a,b\rangle)$ *and for every sequence* $\{\varepsilon_n\}_{n=0}^{\infty}, \varepsilon_n \in J_n$ *there exists a unique solution* y_ε *of problem (1.1) , (1.2) satisfying*

$$y_{\varepsilon_n} \to u \text{ uniformly on } \langle a,b\rangle \quad \text{for } n \to \infty$$

More precisely,

$$y_{\varepsilon_n}(t)=u(t)+O\left(\sqrt{\varepsilon_n}\right) \quad on \ \langle a,b\rangle .$$

Proof. Firstly, we show that

$$y_\varepsilon(t)=\frac{\cos\left[\sqrt{\frac{k}{\varepsilon}}(t-a)\right]\int_a^b \cos\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right]\frac{f(s)}{\varepsilon}ds}{\sqrt{\frac{k}{\varepsilon}}\sin\left[\sqrt{\frac{k}{\varepsilon}}(b-a)\right]}+\int_a^t \frac{\sin\left[\sqrt{\frac{k}{\varepsilon}}(t-s)\right]\frac{f(s)}{\varepsilon}}{\sqrt{\frac{k}{\varepsilon}}}ds \tag{2.1}$$

is a solution of (1.1) , (1.2). Differentiating (2.1) twice, taking into consideration that

$$\frac{d}{dt}\int_a^t H(t,s)f(s)ds=\int_a^t \frac{\partial H(t,s)}{\partial t}f(s)ds \ + \ H(t,t)f(t)$$

we obtain

$$y'_\varepsilon(t)=-\frac{\sqrt{\frac{k}{\varepsilon}}\sin\left[\sqrt{\frac{k}{\varepsilon}}(t-a)\right]\int_a^b \cos\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right]\frac{f(s)}{\varepsilon}ds}{\sqrt{\frac{k}{\varepsilon}}\sin\left[\sqrt{\frac{k}{\varepsilon}}(b-a)\right]}+\int_a^t \frac{\sqrt{\frac{k}{\varepsilon}}\cos\left[\sqrt{\frac{k}{\varepsilon}}(t-s)\right]\frac{f(s)}{\varepsilon}}{\sqrt{\frac{k}{\varepsilon}}}ds \tag{2.2}$$

$$y''_\varepsilon(t)=-\frac{\left(\sqrt{\frac{k}{\varepsilon}}\right)^2\cos\left[\sqrt{\frac{k}{\varepsilon}}(t-a)\right]\int_a^b \cos\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right]\frac{f(s)}{\varepsilon}ds}{\sqrt{\frac{k}{\varepsilon}}\sin\left[\sqrt{\frac{k}{\varepsilon}}(b-a)\right]}-$$

$$-\int_a^t \frac{\left(\sqrt{\frac{k}{\varepsilon}}\right)^2\sin\left[\sqrt{\frac{k}{\varepsilon}}(t-s)\right]\frac{f(s)}{\varepsilon}}{\sqrt{\frac{k}{\varepsilon}}}ds+\frac{f(t)}{\varepsilon} \quad . \tag{2.3}$$

From (2.3) and (2.1) after a little algebraic arrangement, we get

$$y''_\varepsilon=\frac{k}{\varepsilon}(-y_\varepsilon)+\frac{f(t)}{\varepsilon}$$

i.e. y_ε is a solution of differential equation (1.1), and from (2.2) it is easy to verify that this solution satisfies (1.2).

Let $t_0\in\langle a,b\rangle$ be arbitrary, but fixed. Denote by I_1 and I_2 the integrals

$$I_1 = \int_a^b \cos\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right]\frac{f(s)}{\varepsilon}ds$$

$$I_2 = \int_a^{t_0} \sin\left[\sqrt{\frac{k}{\varepsilon}}(t_0-s)\right]\frac{f(s)}{\varepsilon}ds$$

Then

$$y_\varepsilon(t_0) = \frac{\cos\left[\sqrt{\frac{k}{\varepsilon}}(t_0-a)\right]I_1}{\sqrt{\frac{k}{\varepsilon}}\sin\left[\sqrt{\frac{k}{\varepsilon}}(b-a)\right]} + \frac{I_2}{\sqrt{\frac{k}{\varepsilon}}} \ .$$

Integrating I_1 and I_2 by parts, we obtain

$$I_1 = \left| \begin{matrix} h' = \cos\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right] & g = \frac{f(s)}{\varepsilon} \\ h = -\sqrt{\frac{\varepsilon}{k}}\sin\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right] & g' = \frac{f'(s)}{\varepsilon} \end{matrix} \right| =$$

$$= \sqrt{\frac{\varepsilon}{k}}\sin\left[\sqrt{\frac{k}{\varepsilon}}(b-a)\right]\frac{f(a)}{\varepsilon} + \int_a^b \sqrt{\frac{\varepsilon}{k}}\sin\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right]\frac{f'(s)}{\varepsilon}ds$$

$$I_2 = \left| \begin{matrix} h' = \sin\left[\sqrt{\frac{k}{\varepsilon}}(t_0-s)\right] & g = \frac{f(s)}{\varepsilon} \\ h = \sqrt{\frac{\varepsilon}{k}}\cos\left[\sqrt{\frac{k}{\varepsilon}}(t_0-s)\right] & g' = \frac{f'(s)}{\varepsilon} \end{matrix} \right| =$$

$$= \frac{\sqrt{\frac{\varepsilon}{k}}f(t_0)}{\varepsilon} - \sqrt{\frac{\varepsilon}{k}}\cos\left[\sqrt{\frac{k}{\varepsilon}}(t_0-a)\right]\frac{f(a)}{\varepsilon} - \int_a^{t_0}\sqrt{\frac{\varepsilon}{k}}\cos\left[\sqrt{\frac{k}{\varepsilon}}(t_0-s)\right]\frac{f'(s)}{\varepsilon}ds \ .$$

Also

$$y_\varepsilon(t_0) = \frac{f(t_0)}{k} + \frac{\cos\left[\sqrt{\frac{k}{\varepsilon}}(t_0-a)\right]}{\sin\left[\sqrt{\frac{k}{\varepsilon}}(b-a)\right]}\int_a^b \sin\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right]\frac{f'(s)}{k}ds - \int_a^{t_0}\cos\left[\sqrt{\frac{k}{\varepsilon}}(t_0-s)\right]\frac{f'(s)}{k}ds.$$

Now we estimate $y_\varepsilon(t_0) - \frac{f(t_0)}{k}$. We obtain

$$\left| y_\varepsilon(t_0) - \frac{f(t_0)}{k} \right| \le \frac{1}{k\sin\lambda} \left| \int_a^b \sin\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right] f'(s)ds \right| + \frac{1}{k} \left| \int_a^{t_0} \cos\left[\sqrt{\frac{k}{\varepsilon}}(t_0-s)\right] f'(s)ds \right| . \qquad (2.4)$$

The integrals in (2.4) converge to zero for $\varepsilon = \varepsilon_n, \varepsilon_n \in J_n, n \to \infty$.

Indeed, with respect to assumption on f, we may integrate by parts in (2.4). Thus,

$$\int_a^b \sin\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right] f'(s)ds = \left[\sqrt{\frac{\varepsilon}{k}} \cos\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right] f'(s)\right]_a^b - \int_a^b \sqrt{\frac{\varepsilon}{k}} \cos\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right] f''(s)ds \le$$

$$\le \sqrt{\frac{\varepsilon}{k}} \left(|f'(a)| + |f'(b)| + \left| \int_a^b \cos\left[\sqrt{\frac{k}{\varepsilon}}(b-s)\right] f''(s)ds \right| \right) \le$$

$$\le \sqrt{\frac{\varepsilon}{k}} \left\{ |f'(a)| + |f'(b)| + \sqrt{\frac{\varepsilon}{k}} \left(|f''(a)| + \mu_2(b-a) \right) \right\} \qquad (2.5)$$

and

$$\int_a^{t_0} \cos\left[\sqrt{\frac{k}{\varepsilon}}(t_0-s)\right] f'(s)ds = \left[-\sqrt{\frac{\varepsilon}{k}} \sin\left[\sqrt{\frac{k}{\varepsilon}}(t_0-s)\right] f'(s)\right]_a^{t_0} + \int_a^{t_0} \sqrt{\frac{\varepsilon}{k}} \sin\left[\sqrt{\frac{k}{\varepsilon}}(t_0-s)\right] f''(s)ds \le$$

$$\le \sqrt{\frac{\varepsilon}{k}} \left(|f'(a)| + \left| \int_a^{t_0} \sin\left[\sqrt{\frac{k}{\varepsilon}}(t_0-s)\right] f''(s)ds \right| \right) \le$$

$$\le \sqrt{\frac{\varepsilon}{k}} \left\{ |f'(a)| + \sqrt{\frac{\varepsilon}{k}} \left(\mu_1 + |f''(a)| + \mu_2(b-a) \right) \right\}, \qquad (2.6)$$

where $\mu_1 = \sup_{t \in \langle a,b \rangle} |f''(t)|$ *and* $\mu_2 = \sup_{t \in \langle a,b \rangle} |f'''(t)|$.

Substituting (2.5) and (2.6) into (2.4), we obtain an a priori estimate of solutions of (1.1), (1.2) for all $t_0 \in \langle a,b \rangle$ of the form

$$\left| y_\varepsilon(t_0) - \frac{f(t_0)}{k} \right| \le \frac{1}{k\sin\lambda} \sqrt{\frac{\varepsilon}{k}} \left\{ |f'(a)| + |f'(b)| + \sqrt{\frac{\varepsilon}{k}} \left(|f''(a)| + \mu_2(b-a) \right) \right\} +$$

$$+ \frac{1}{k} \sqrt{\frac{\varepsilon}{k}} \left\{ |f'(a)| + \sqrt{\frac{\varepsilon}{k}} \left(\mu_1 + |f''(a)| + \mu_2(b-a) \right) \right\} \qquad (2.7)$$

Because the right side of the inequality (2.7) is independent on t_0 the convergence is uniformly on $\langle a,b \rangle$. Theorem holds.

Remark. As remark we conclude that in the case $|f'(a)| = |f'(b)| = 0$, the convergence rate is $O(\varepsilon_n), \varepsilon_n \in J_n$, as follows from (2.7).

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Conclusion

In our contribution, we determined a convergence rate of the solutions of a certain class of the singularly perturbed differential equations subject to Neumann boundary conditions to the solution of a reduced problem as a small parameter ε at highest derivative tends to zero.

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