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ON STABILITY OF CERTAIN CLASS OF 3-RD ORDER NONLINEAR CONTROL SYSTEMS

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Abstract

The paper deals with a certain class of nonautonomous ordinary third-order nonlinear differential equations $L_3 y = f\left(t, L_0 y, L_1 y, L_2 y\right)$ *with quasi-derivatives. A criterion of asymptotic stability in Liapunov sense as well as a criterion of instability in Liapunov sense is derived. The results are illustrated by two examples.*

Key words

nonautonomous nonlinear differential equation; 3-rd order; quasi-derivative; stability in Liapunov sense; asymptotic stability in Liapunov sense, instability in Liapunov sense

Introduction

In the frame of Control Theory, Palumbiny in (5) derived stability criteria of certain class of third-order nonlinear differential equations with so called quasi-derivatives (see the text below). Then we stated such results for autonomous nonlinear 3-rd equations (see (7)). The main aim of the presented article is to establish similar criteria for another class of third-order nonlinear differential equations with quasi-derivatives. Our paper deals with a criterion of asymptotic stability as well as instability, both in Liapunov sense, of a null solution 0 of the nonautonomous nonlinear third-order differential equations with the quasi-derivatives

[L] $$L_3 y = f\left(t, L_0 y, L_1 y, L_2 y\right),$$

where (the prime means a derivative owing to the variable t)

$$L_0 y(t) = y(t),$$
$$L_1 y(t) = p_1(t)\left(L_0 y(t)\right)',$$
$$L_2 y(t) = p_2(t)\left(L_1 y(t)\right)',$$
$$L_3 y(t) = \left(L_2 y(t)\right)',$$

$p_i(t),\, i=1,\,2$ are real-valued continuous functions defined on an open interval (b,∞), $f(t, y_1, y_2, y_3)$ is real-valued and continuous up to all 2-nd order partial derivatives of the function f owing to $y_k,\ k=1,2,3.$ A symbol E denotes the set of all real numbers. The symbol $\mathbf{o}$ means the vector $(0,0,0)$. The symbol 0 is, according to a situation, a real number null or a real null function. The terms $L_k y(t),\ k=0,1,2,3$ are so called *k-th quasi-derivatives* of a function $y(t)$.
We note that we shall use a matrix norm of the form $\| \{a_{ij}\}_{i,j} \| = \sum_{i,j} |a_{ij}|$. A set $[-1,1]$ is a closed interval with bounds $-1,\,1$. An important contribution of the article consists in the fact that there are more control parametres in [L]. These parametres are the functions $p_i(t),\, i=1,2$ which enable an user a better control of considered processes described by the equation [L].

Remark 1. The equation [L] can be expressed more detailed as

[M] $$\left(p_2(t)\left(p_1(t)y'\right)'\right)' = f\left(t, y, p_1(t)y', p_2(t)\left(p_1(t)y'\right)'\right).$$

Now we determine notions concerning the stability resp. instability in Liapunov sense. Let us consider a general differential system of the first order

[S] $$\begin{aligned} y_1' &= f_1(t, y_1, y_2, y_3) \\ y_2' &= f_2(t, y_1, y_2, y_3) \\ y_3' &= f_3(t, y_1, y_2, y_3) \end{aligned}$$

Assumption. Let the system [S] be expressed in a matrix form $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$. Throughout the paper we shall assume an existence of a number b (real or $-\infty$) and an area $H \subset E^3$, $\mathbf{o} \in H$ such that the function f is continuous on a set $G = (b,\infty) \times H$ and for every point $(\tau, \mathbf{k}) \in G$ the following Cauchy problem

[1] $$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}),\ \mathbf{y}(\tau) = \mathbf{k},$$

admits the only solution. We also assume $\mathbf{f}(t, \mathbf{o}) = \mathbf{o}$ for all $t > b$, i.e. the Cauchy problem [1] admits for $\mathbf{k} = \mathbf{o}$ the trivial solution defined by a formula $\mathbf{o}(t) = \mathbf{o}$ for all $t > b$.

Definition 1. We say that the trivial solution $\mathbf{o}$ of the *system* [S] *is stable in Liapunov sense*, if for every $\tau > b$ and every $\varepsilon > 0$ there exists $\delta = \delta(\tau, \varepsilon) > 0$ such that for every initial values $\mathbf{k} \in H, \|\mathbf{k}\| < \delta$ and for all $t \geq \tau$ it holds that the solution $\mathbf{u}(t, \tau, \mathbf{k})$ of the Cauchy problem [1] fulfils the following inequality

[2] $$\| \mathbf{u}(t,\tau,\mathbf{k}) \| < \varepsilon .$$

Otherwise, the trivial solution $\mathbf{o}$ is instable in Liapunov sense.

Definition 2. We say that the trivial solution $\mathbf{o}$ of the *system* [S] *is asymptotically stable in Liapunov sense*, if the trivial solution $\mathbf{o}$ is stable in Liapunov sense and there exists a real number $\Delta > 0$ such that for all $\mathbf{k} \in H, \| \mathbf{k} \| < \Delta$ and every $\tau > b$ it holds that

$$\lim_{t\to\infty} \| \mathbf{u}(t,\tau,\mathbf{k}) \| = 0.$$

Definition 3. A special type [T] of the system [S] of the form

[T]
$$\begin{aligned} y_1' &= y_2 / p_1(t) \\ y_2' &= y_3 / p_2(t) \\ y_3' &= f(t, y_1, y_2, y_3) \end{aligned}$$

is called a *competent system to the equation* [L].

Remark 2. We recall an important property of the system [T] which consists in a fact that a function $u(t)$ is a solution of [L] if and only if a vector $(u(t), L_1u(t), L_2u(t))$ is a solution of [T].

Definition 4. Let [L] be such an equation that the function 0 is a solution of [L] on (b,∞). Then, according to Remark 2, the vector $(0,0,0)$ is a solution of [T]. We say 0 is a *stable solution of* [L] *in Liapunov sense*, if $(0,0,0)$ is a stable solution of [T] in Liapunov sense. Otherwise, 0 is an *instable solution of* [L] *in Liapunov sense*.

Definition 5. Let [L] be such an equation that 0 is a solution of [L] on (b,∞). Then, according to Remark 2, the vector $(0,0,0)$ is a solution of [T]. We say 0 *is an asymptotically stable solution of* [L] *in Liapunov sense*, if $(0,0,0)$ is an asymptotically stable solution of [T] in Liapunov sense.

The main aim of the paper is to establish the criteria, which assure the asymptotic stability as well as instability of the null solution 0 of the equation [L]. If we put $p_k(t) = 1$ on (b,∞), $k = 1,2$ in [L], we obtain a differential equation with classic derivatives. We note that the functions $p_k(t)$, $k = 1,2$ are not, in general, assumed to be differentiable. From this it follows that we cannot use on [L] stability criteria derived for nonlinear differential equations with classic derivatives.

Auxiliary assertions

Now we introduce some auxiliary assertions which are significant for our considerations. The first of them is the special case of the wellknown Hurwitz criterion when $n = 3$ (see (2), Chapter 9 or (3)):

Theorem 1. Let us consider a polynomial

[3] $$b_3 s^3 + b_2 s^2 + b_1 s + b_0,$$

where $b_i,\ i = 0,1,2,3$ are real numbers such that $b_0 > 0,\ b_3 \neq 0.$ Then all zeros of the polynomial [3] admit negative real parts if and only if it holds that

[4] $$b_1 > 0,$$

[5] $$b_1 b_2 - b_0 b_3 > 0,$$

[6] $$b_1 b_2 b_3 - b_0 b_3^2 > 0.$$

The second assertion deals with asymptotic criteria of stability in Liapunov sense for systems of differential equations of the first order (see (1), Chapter 13 or (4)):

Theorem 2. Let us consider a system of differential equations of the first order expressed in the following matrix form

[7] $$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathrm{t})\mathbf{x} + \mathbf{g}(\mathrm{t}, \mathbf{x}),\ \mathbf{g}(t, \mathbf{o}) = \mathbf{o},$$

where $\mathbf{A}$ is a real constant square matrix, $\mathbf{B}(t)$ is a real square matrix depending on t only, such that

[8] $$\lim_{t \to \infty} \mathbf{B}(t) = \mathbf{0},$$

where $\mathbf{0}$ is a null matrix and $\mathbf{g}$ a real vector function continuous on an area $(b, \infty) \times H,$ where $b \in E,$ satisfying a condition

[9] $$\lim_{\|\mathbf{x}\| \to 0} \frac{\| \mathbf{g}(t, \mathbf{x}) \|}{\| \mathbf{x} \|} = 0 \text{ uniformly for all } t \geq b.$$

Then:

[i] If all eigenvalues of $\mathbf{A}$ have negative real parts, then the trivial solution $\mathbf{o}$ of [7] is asymptotically stable in Liapunov sense.

[ii] If at least one of eigenvalues of $\mathbf{A}$ has a positive real part, then the trivial solution $\mathbf{o}$ of [7] is instable in Liapunov sense.

Theorem 3. Let $\lim_{\|\mathbf{x}\| \to 0} \frac{\| \mathbf{g}_1(t, \mathbf{x}) \|}{\| \mathbf{x} \|} = 0$ uniformly for all $t \geq b$ as well as let $\lim_{\|\mathbf{x}\| \to 0} \frac{\| \mathbf{g}_2(t, \mathbf{x}) \|}{\| \mathbf{x} \|} = 0$ uniformly for all $t \geq b.$ Then $\lim_{\|\mathbf{x}\| \to 0} \left(\frac{\| \mathbf{g}_1(t, \mathbf{x}) \|}{\| \mathbf{x} \|} + \frac{\| \mathbf{g}_2(t, \mathbf{x}) \|}{\| \mathbf{x} \|} \right) = 0$ uniformly for all $t \geq b$.

Proof. The proof immediately follows from the fact, that a norm of sum of two functions is less than or equal to sum of norms of these functions for any $t \geq b$.

Results

Now we shall prove the first main result of the paper – the criterion of asymptotic stability of the null solution 0 in Liapunov sense of the differential equation [L]:

Theorem 4. Let us consider the differential equation [L] such that

[a] $$\lim_{t\to\infty} p_i(t) = a_i > 0,\ i = 1,2.$$

Let $\mathbf{w} = (w_1, w_2, w_3)$. If it holds that

[b] $$\lim_{t\to\infty} f(t, w_1, w_2, w_3) = f(\infty, w_1, w_2, w_3),$$

[c] $$\frac{\partial f}{\partial w_1}(\infty,\ \mathbf{o}) < 0,\ \frac{\partial f}{\partial w_3}(\infty,\ \mathbf{o}) < 0,\ a_1 \frac{\partial f}{\partial w_2}(\infty,\ \mathbf{o}) \frac{\partial f}{\partial w_3}(\infty,\ \mathbf{o}) + \frac{\partial f}{\partial w_1}(\infty,\ \mathbf{o}) > 0,$$

[d] $$\lim_{\|\mathbf{w}\|\to 0} \frac{|\, f(t, w_1, w_2, w_3) - f(\infty, w_1, w_2, w_3)\,|}{\|\,\mathbf{w}\,\|} = 0 \ \text{ uniformly for } \ t \in (b, \infty),$$

then the null solution 0 of [L] on (b,∞) is asymptotically stable in Liapunov sense.

Proof. The equation [L] can be expressed in the form

$$L_3 y = f\left(\infty, L_0 y, L_1 y, L_2 y\right) + [\, f(t, L_0 y, L_1 y, L_2 y) - f(\infty, L_0 y, L_1 y, L_2 y)].$$

The null solution 0 is, according to Definition 5, asymptotically stable in Liapunov sense, if the solution $\mathbf{o}$ of the system [T] is asymptotically stable in Liapunov sense, where [T] is expressed in the form [U], where

[U] $$\mathbf{w}' = \mathbf{A}\mathbf{w} + \mathbf{B}(t)\mathbf{w} + \mathbf{g}(t,\mathbf{w}), \qquad \mathbf{g}(t,\mathbf{o}) = \mathbf{o},$$

and

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \qquad \mathbf{w}' = \begin{bmatrix} w_1' \\ w_2' \\ w_3' \end{bmatrix}, \qquad \mathbf{o} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0, & \dfrac{1}{a_1}, & 0 \\ 0, & 0, & \dfrac{1}{a_2} \\ \dfrac{\partial f}{\partial w_1}(\infty,\mathbf{o}), & \dfrac{\partial f}{\partial w_2}(\infty,\mathbf{o}), & \dfrac{\partial f}{\partial w_3}(\infty,\mathbf{o}) \end{bmatrix}$$

$$\mathbf{B}(t) = \begin{bmatrix} 0, & \dfrac{1}{p_1(t)} - \dfrac{1}{a_1}, & 0 \\ 0, & 0, & \dfrac{1}{p_2(t)} - \dfrac{1}{a_2} \\ 0, & 0, & 0 \end{bmatrix}, \qquad \mathbf{g}(t,\mathbf{w}) = \mathbf{g}_1(t,\mathbf{w}) + \mathbf{g}_2(t,\mathbf{w}),$$

$$\mathbf{g}_1(t,\mathbf{w}) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{2}\left(\mathrm{w}_1 \dfrac{\partial}{\partial w_1} + \mathrm{w}_2 \dfrac{\partial}{\partial w_2} + \mathrm{w}_3 \dfrac{\partial}{\partial w_3} \right)^2 f(\infty, \Theta\mathbf{w})) \end{bmatrix}, \qquad \mathbf{g}_2(t,\mathbf{w}) = \begin{bmatrix} 0 \\ 0 \\ f(t,\mathbf{w}) - f(\infty,\mathbf{w}) \end{bmatrix}$$

,

where we used on the function $\mathbf{g}_1(t,\mathbf{w})$ which does not depend on the variable t the Taylor's theorem with the remainder in the Lagrange's form as $k = 2,\ 0 < \Theta < 1.$ Then

$$0\le\frac{\|\mathbf{g}_1(t,\mathbf{w})\|}{\|\mathbf{w}\|}=\frac{|0|+|0|+\left|\frac{1}{2}\left(\mathrm{w}_1\frac{\partial}{\partial w_1}+\mathrm{w}_2\frac{\partial}{\partial w_2}+\mathrm{w}_3\frac{\partial}{\partial w_3}\right)^2 f(\infty,\Theta\mathbf{w})\right|}{|w_1|+|w_2|+|w_3|}\le$$

[10] $$\le\frac{1}{|w_1|+|w_2|+|w_3|}\left(\left|\frac{1}{2}\frac{\partial^2 f}{\partial w_1^2}(\infty,\Theta\mathbf{w})w_1^2\right|+\left|\frac{1}{2}\frac{\partial^2 f}{\partial w_2^2}(\infty,\Theta\mathbf{w})w_2^2\right|+\left|\frac{1}{2}\frac{\partial^2 f}{\partial w_3^2}(\infty,\Theta\mathbf{w})w_3^2\right|+\right.$$
$$\left.+\left|\frac{\partial^2 f}{\partial w_1\partial w_2}(\infty,\Theta\mathbf{w})w_1w_2\right|+\left|\frac{\partial^2 f}{\partial w_1\partial w_3}(\infty,\Theta\mathbf{w})w_1w_3\right|+\left|\frac{\partial^2 f}{\partial w_2\partial w_3}(\infty,\Theta\mathbf{w})w_2w_3\right|\right)$$

Without loss of generality, we can assume that $(w_1,w_2,w_3)\in[-1,1]^3$. From this and from the continuity of all 2-nd order partial derivatives of the function f it follows that there exists a positive real constant K such that

[11] $$\left|\frac{1}{2}\frac{\partial^2 f}{\partial w_k^2}(\infty,\Theta\mathbf{w})\right|\le K,\ k=1,\,2,\,3,\qquad\left|\frac{\partial^2 f}{\partial w_i\partial w_j}(\infty,\Theta\mathbf{w})\right|\le K,\ (i,j)=(1,2),\,(1,3),\,(2,3)$$

for all $(w_1,w_2,w_3)\in[-1,1]^3$. If we use the estimations [11] in the formula [10], we obtain

$$0\le\frac{\|\mathbf{g}_1(t,\mathbf{w})\|}{\|\mathbf{w}\|}\le\frac{K}{|w_1|+|w_2|+|w_3|}\left(|w_1^2|+|w_2^2|+|w_3^2|+|w_1w_2|+|w_1w_3|+|w_2w_3|\right)=$$
$$=K\left(\frac{|w_1|+|w_2|}{|w_1|+|w_2|+|w_3|}|w_1|+\frac{|w_2|+|w_3|}{|w_1|+|w_2|+|w_3|}|w_2|+\frac{|w_1|+|w_3|}{|w_1|+|w_2|+|w_3|}|w_3|\right)\le K\left(|w_1|+|w_2|+|w_3|\right)=K\|\mathbf{w}\|.$$

The squeeze theorem from Limit Theory yields that $\frac{\|\mathbf{g}_1(t,\mathbf{w})\|}{\|\mathbf{w}\|}$ converges to zero as $\|\mathbf{w}\|\to 0.$ This convergence is uniform because the term $K\left(|w_1|+|w_2|+|w_3|\right)$ does not explicitly depend on the variable t. From [b] as well as [d] we obtain $\frac{\|\mathbf{g}_2(t,\mathbf{w})\|}{\|\mathbf{w}\|}$ uniformly converges to zero as $\|\mathbf{w}\|\to 0.$ Then Theorem 3 yields that [9] hold. We can easily observe a validity of the conditions [7], [8] in Theorem 2. The validity of condition [a], [c] assure that the conditions [4], [5], [6] hold in Theorem 1, where characteristic polynomial of the matrix $\mathbf{A}$ is

$$s^3-\frac{\partial f}{\partial w_3}(\infty,\,\mathbf{o})s^2-\frac{1}{a_2}\frac{\partial f}{\partial w_2}(\infty,\,\mathbf{o})s-\frac{1}{a_1a_2}\frac{\partial f}{\partial w_1}(\infty,\,\mathbf{o}).$$

Then Theorem 1 yields that all the eigenvalues of $\mathbf{A}$ have the negative real parts. Consequently, Theorem 2, the part [i] as well as Definition 5 yield the required stability of the null solution 0 of [L].

Example 1. Let us consider the differential equation [L], where $p_1(t)=2+\frac{1}{t}$, $p_2(t)=3-\frac{1}{t}$ and

$$L_3y=f\left(t,L_0y,L_1y,L_2y\right)=\frac{y^4}{1+(L_2y)^2}\left(1+\frac{2}{t}\right)-y-2L_1y-3L_2y.$$

Then

$$f(\infty, w_1, w_2, w_3) = \lim_{t\to\infty} f(t, w_1, w_2, w_3) = \frac{y^4}{1+(L_2 y)^2} - y - 2L_1 y - 3L_2 y.$$

It is obvious that Assumption, mentioned after Remark 1, hold for $b = 2.$ An easy computing yields that the nonlinear differential equation [L] admits the null solution 0 as well as $a_1 = 2,\ a_2 = 3,\ \frac{\partial f}{\partial w_1}(\infty,\ \mathbf{o}) = -1,\ \frac{\partial f}{\partial w_2}(\infty,\ \mathbf{o}) = -2,\ \frac{\partial f}{\partial w_3}(\infty,\ \mathbf{o}) = -3$. From this immediately follows the validity of [a], [b] and [c] in Theorem 4. Then

$$0 \le \frac{|\, f(t, w_1, w_2, w_3) - f(\infty, w_1, w_2, w_3)\,|}{\|\,\mathbf{w}\,\|} = \frac{\left|\frac{w_1^{\,4}}{1+w_3^{\,2}} \cdot \frac{2}{t}\right|}{|\,w_1\,| + |\,w_2\,| + |\,w_3\,|} \le \frac{\left|w_1^{\,4}\right| \cdot \max\limits_{t\ge 2}\left\{\frac{2}{t}\right\}}{|\,w_1\,| + |\,w_2\,| + |\,w_3\,|} =$$

$$= \frac{\left|w_1^{\,4}\right| \cdot 1}{|\,w_1\,| + |\,w_2\,| + |\,w_3\,|} = \frac{|w_1|}{|\,w_1\,| + |\,w_2\,| + |\,w_3\,|} \cdot \left|w_1^{\,3}\right| \le \left|w_1^{\,3}\right|.$$

From this immediately follows $\lim\limits_{\|\mathbf{w}\|\to 0} \frac{|\, f(t, w_1, w_2, w_3) - f(\infty, w_1, w_2, w_3)\,|}{\|\,\mathbf{w}\,\|} = 0$ uniformly for $t \in (b, \infty)$ because of the expression $\left|w_1^{\,3}\right|$ does not depend on the variable t. Consequently, the condition [d] in Theorem 4 holds. Then the last mentioned theorem yields required stability of the null solution 0 of the equation [L].

Now we shall prove the second main result of the paper – the criterion of instability of the null solution 0 in Liapunov sense of the differential equation [L]:

Theorem 5. Let us consider the differential equation [L] such that

[a] $$\lim_{t\to\infty} p_i(t) = a_i > 0,\ i = 1, 2.$$

If it hold that

[b] $$\lim_{t\to\infty} f(t, w_1, w_2, w_3) = f(\infty, w_1, w_2, w_3),$$

[c'] at least one real part of zeros of $s^3 - \frac{\partial f}{\partial w_3}(\infty,\ \mathbf{o}) s^2 - \frac{1}{a_2}\frac{\partial f}{\partial w_2}(\infty,\ \mathbf{o}) s - \frac{1}{a_1 a_2}\frac{\partial f}{\partial w_1}(\infty,\ \mathbf{o})$ is positive,

[d] $$\lim_{\|\mathbf{w}\|\to 0} \frac{|\, f(t, w_1, w_2, w_3) - f(\infty, w_1, w_2, w_3)\,|}{\|\,\mathbf{w}\,\|} = 0 \text{ uniformly for } t \in (b, \infty),$$

then the null solution 0 of the equation [L] is instable in Liapunov sense.

Proof. The null solution 0 of [L] is, according to Definition 4, instable in Liapunov sense, if the solution $(0,0,0)$ of the system [U] is instable in Liapunov sense. By the same way as in the proof of Theorem 4, it can by proved the validity of [9]. We can easily observe the validity of the conditions [7], [8] in Theorem 2. Then the last mentioned Theorem, the part [ii] yields the required instability of the null solution 0 of [L].

Example 2. Let us consider the equation [L], where $p_1(t)=2+\frac{1}{t}$, $p_2(t)=3-\frac{1}{t}$ and

$$L_3 y=\frac{y^6}{4+(L_1 y)^2}\left(1+\frac{2}{t}\right)+(y-2L_1 y-3L_2 y).$$

Then

$$f(\infty,w_1,w_2,w_3)=\lim_{t\to\infty} f(t,w_1,w_2,w_3)=\frac{y^6}{1+(L_2 y)^2}-y-2L_1 y-3L_2 y.$$

It is obvious that Assumption, mentioned after Remark 1, hold for $b=2.$ An easy computing yields that the function 0 is the null solution of the differential equation [L] as well as $a_1=2,\ a_2=3,\ \frac{\partial f}{\partial w_1}(\infty,\mathbf{o})=1,\ \frac{\partial f}{\partial w_2}(\infty,\mathbf{o})=-2,\ \frac{\partial f}{\partial w_3}(\infty,\mathbf{o})=-3.$ From this immediately follows the validity of [a] and [d]. By the same way as in Example 1 it can be proved the property [c']. Then the characteristic polynomial of $\mathbf{A}$ is

$$h(s)=s^3+3s^2+\frac{2}{3}s-\frac{1}{6}.$$

There are two possibilities only: 1) $h(s)$ admits three real zeros. Then their product is equal to $1/6$. It means, that all these zeros differ null. If all these zeros were negative, then their product would be negative, which is a contradiction. 2) $h(s)$ admits one real zero a and two complex zeros $b\pm ci.$ Then their product $a(b^2+c^2)$ equals to $1/6$ again. If $a\le 0,$ then this product would be nonpositive. Thus $a>0.$ Then, owing to Theorem 5, the function 0 is an instable solution of [L] in Liapunov sense.

Conclusion

The foregoing results can be used for ordinary nonlinear differential equations with quasi-derivatives. Especially, when the functions $p_1(t),\ p_2(t)$ are not differentiable on a considered interval, where classical stability criteria cannot be used. The differential equations in applications where the quasi-derivatives have been occurred are, for example, the differential equations describing a stationary distribution of temperature in a wall of a circle tube as well as the differential equations of an equilibrium state of a straight mass bar. For more details see (6).

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