ELESTROSTATIC FIELD DISPERSION IN PLATE CONDENSOR CAUSED BY ELECTRODES DISPARITY

ROZPTYL ELEKTROSTATICKÉHO POĽA V ROVINNOM KONDENZÁTORE SPÔSOBENÝ DISPARITOU ELEKTRÓD

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Abstract

Special case of electrostatic field spatial distribution in homogeneous dielectric material inserted between electrodes of circular plate condensor with unequal diameter was calculated. We was especially interested in axially symmetric field dissipation in the area behind the edge of the circular electrode with smaller radius. Generally the typical case of Sturm-Liouville boundary value problem called Bessel`s differential equation arises in calculation of the scalar potential with axial symmetry. We found the particular solution of this equation which define the Bessel functions and expressed the potential of electrostatic fields in material. Consequently we considered a limit case of extremely slim condensor and investigated decrease of the field potential in the area behind the edge of smaller electrode by maens of asymptotic forms of Bessel functions. Uncertainty of capacity determination can be evaluated by means of obtained results.

V príspevku sa venujeme problematike výpočtu priestorového rozloženia elektrostatického poľa v homogénnom dielektriku vloženom medzi elektródy rovinného kondenzátora tvaru kruhových dosiek s nerovnakým polomerom. Zaujímame sa predovšetkým o osovo symetrický prípad rozptylu elektrostatického poľa v oblasti za hranou elektródy s menším polomerom. Výpočet skalárneho potenciálu v prípade osovo symetrických elektród vo všeobecnosti vyúsťuje do problému riešenia Besselovej diferenciálnej rovnice. Našli sme partikulárne riešenie

tejto rovnice v tvare Besselových funkcií a vyjadrili potenciál elektrostatického poľa v dielektriku. Uvážili sme limitný prípad extrémne tenkého kondenzátora a vyšetrovali pokles potenciálu poľa v oblasti za hranou menšej elektródy pomocou asymptotických Besselových funkcií. Na základe uvedených výsledkov bol uskutočnený odhad neurčitosti v stanovení kapacity.

Key words

Laplace equation, Bessel`s differential equation, Bessel functions,

Laplaceova rovnica, Besselova diferenciálna rovnica, Besselove funkcie,

INTRODUCTION

In presented paper we deal with calculation of electrostatic field in dielectric medium inserted between two circular shaped electrodes with different diameters. We consider a cylindric sample made of investigated dielectric material with radius R_1 and very thin heigth h ($h << R_2$) inserted between two parallel coaxial metal plates with radius R_1 a R_2 ($R_2 < R_1$). Metal plates are connected to voltage U and charged consequently (scheme is shown in fig.1). A same quantum of charges Q with opposed signs are accumulated on the metal plates. Metal plate with radius R_1 is charged on potential φ_1 and metal plate with radius R_2 is charged on potential φ_2. We are interested in electrostatic field potential φ dependence on distance r from the axis of both plates measured on the sample surface mainly in area $R_2 \le r \le R_1$ (for $z = h$ according fig.1). Dispersion of the fieldcan be observed in mentioned area. In next paper we investigate the electrostatic field in this area.

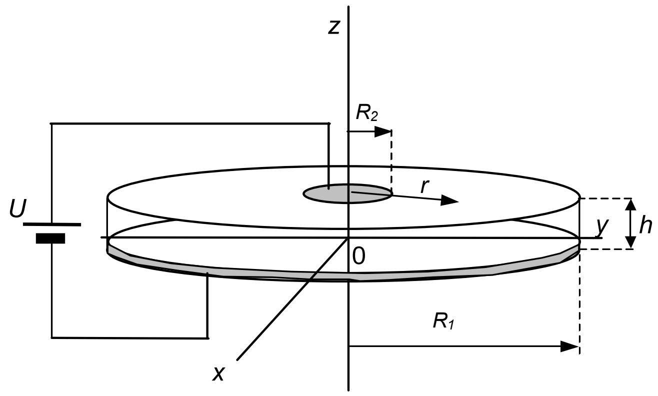


fig.1

ELESTROTATIC FIELD IN CONDENSOR DIELECTRIC

It is possible to consider Laplace equation [1, 2] for charges free electrostatic field at investigation of potential spatial changes in dielectric media. Mentioned equation can be written as:

$$\Delta \varphi = 0 , \tag{1}$$

If we assume axial symmetry of homogeneous dielectric media (see fig.1) it is advantageous to transform mentioned equation (1) to cylindrical coordinates:

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{2}$$

and consider the fact:

$$\frac{\partial^2 \varphi}{\partial \theta^2} = 0 \,. \tag{3}$$

Consequently the scalar potential in medium $\varphi(r;z)$ can be find as a solution of following equation:

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \,. \tag{4}$$

If we expect a separable solution:

$$\varphi(r;z) = \Phi(r)\,Y(z) \,, \tag{5}$$

it results from equation (4) that solution components $\Phi(r)$ and $Y(z)$ must obey equations:

$$\frac{\partial^2 \Phi(r)}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi(r)}{\partial r} = \lambda\,\Phi(r) \,, \tag{6}$$

$$\frac{\partial^2 Y(z)}{\partial z^2} = -\lambda\,Y(z) \,, \tag{7}$$

where λ is constant that can be positive, negative or it can be equal zero. In following text we investigate mentioned events.

a) In case if $\lambda = 0$ it is possible the solution of equation (6) to find by substitution:

$$f(r) = \frac{\partial \Phi(r)}{\partial r} \,. \tag{8}$$

It is easy to show that:

$$f(r) = e^{-C_0}\, e^{-ln(r)} = \frac{A_0}{r} \tag{9}$$

and:

$$\Phi_0(r) = A_0 \, ln\, r + B_0 \,. \tag{10}$$

Integration constants A_0 a B_0 depend on boundary conditions. Function $Y(z)$ in case if $\lambda = 0$ can be determined very easy from equation (7):

$$Y_0(z) = C_0\, z + D_0 \,, \tag{11}$$

where C_0 and D_0 are integration constants. Consequantly in mention case ($\lambda = 0$) the particular solution of differential equation (4) can be written in following form:

$$\varphi_0(r;z) = K_0\, z\, ln\, r + L_0\, ln\, r + M_0\, z + N_0 \,. \tag{12}$$

Integration constants K_0, L_0, M_0 a N_0 must be determined by means of boundary conditions.

b) In case if $\lambda > 0$ we can write:

$$\lambda = k^2 \,. \tag{13}$$

Solution of equation (7) can be find in form:

$$Y_p(z) = Y_1\, sin(kz + \alpha) \,, \tag{14}$$

in that case. Equation (6) can be transformed to special case of modified Bessel`s differential equation:

$$x^2 \frac{\partial^2 \Phi}{\partial x^2} + x\frac{\partial \Phi}{\partial x} - x^2\,\Phi = 0 \tag{15}$$

by means of substitution:

$$x = k\,r\,.\tag{16}$$

Solution of (6) can be determined as follows:

$$\Phi_p(r) = A_2\,I_0(k\,r) + B_2\,K_0(k\,r)\,.\tag{17}$$

in that case, where I_0 and K_0 are modified Bessel`s functions [3, 4] of zero order:

$$I_0(k\,r) = \sum_{m=0}^{\infty} \frac{1}{(m!)^2}\left(\frac{k\,r}{2}\right)^{2m}\tag{18}$$

$$K_0(k\,r) = -I_0(k\,r)\left\{\eta + ln\left(\frac{k\,r}{2}\right)\right\} + \frac{2}{1}I_2(k\,r) + \frac{2}{2}I_4(k\,r) + \frac{2}{3}I_6(k\,r) + \ldots\quad .\tag{19}$$

Particular solution of equation (4) is:

$$\varphi_p^{(1)} = Y^0\left[A_2\,I_0(k\,r) + B_2 K_0(k\,r)\right] sin(kz + \alpha)\tag{20}$$

and general solution can be find as:

$$\varphi^{(1)}(r;z) = Y^0\left[A_2\,I_0(k\,r) + B_2\,K_0(k\,r)\right] sin(k\,z + \alpha) + K_0\,z\,ln\,r + L_0\,ln\,r + M_0\,z + N_0\tag{21}$$

in case of the $\lambda > 0$ consequently.

c) In case of $\lambda < 0$ we can write:

$$\lambda = -k^2\,.\tag{22}$$

and solution of equation (7) is:

$$Y(x) = A_3\,e^{kz} + B_3\,e^{-kz}\,,\tag{23}$$

where A_3 a B_3 are integration constants.
Equation (6) can be transformed as follows:

$$x^2\,\frac{\partial^2\,\Phi}{\partial x^2} + x\,\frac{\partial \Phi}{\partial x} + x^2\,\Phi = 0\tag{24}$$

by means of substitution (16) in mentioned case. Equation (24) is special case of Bessel`s differential equation [4, 5] and solution of (6) can be written in form:

$$\Phi_p^{(2)}(x) = K_2\,J_0(x) + M_2\,Y_0(x)\,,\tag{25}$$

where:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}\tag{26}$$

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left\{C + ln\left(\frac{x}{2}\right)\right\} + \frac{2}{1}J_2(x) - \frac{2}{2}J_4(x) + \frac{2}{3}J_6(x) + \ldots\right]\tag{28}$$

are Bessel`s fonctions of zero order.
General solution of equation (6) is:

$$\varphi^{(2)}(r;z) = \left[K_2\,J_0(k\,r) + M_2\,Y_0(k\,r)\right]\left(A_3\,e^{kz} + B_3\,e^{-kz}\right) + K_0\,z\,ln\,r + L_0\,ln\,r + M_0\,z + N_0\tag{29}$$

in case of $\lambda < 0$.

APPLICATION OF BOUNDARY CONDITIONS

1. Spatial changes of scalar potential is determined by function $\varphi^{(I.)}(r;z)$ in area $0 \le r \le R_2$, $z \le 0 \le h$ (area I). Function $\varphi^{(I.)}(r;z)$ must obey following conditions:

$$\varphi^{(I.)}(r;0) = \varphi_1\,,\,\varphi^{(I.)}(r;h) = \varphi_2\tag{30}$$

after the charging of condensor.

The solution (21) can be in accordance with boundary conditions (30). Spatial change of electrostatic potential in investigated area is possible to write in the form (21) and the solution following conditions must satisfy:

$$sin(k\,h+\alpha)=0\,,\ sin(k\,0+\alpha)=0\,. \tag{31}$$

It follows from (31):

$$k\,h+\alpha=n\,\pi\,,\ k\,h+\alpha=n\,\pi \tag{32}$$

where *n* a *m* are integral numbers (0, ±1, ±2, ±3,...). Taking into account the fact that there are no reason that any part of solution is periodical it is possible to appoint *m* = 0 and *n* = 1 and:

$$\alpha=0\,,\ k=\frac{\pi}{h}\,. \tag{33}$$

In addition the boundary conditions (30) can be satisfied only in case:

$$K_0=0\,,\ L_0=0\,. \tag{34}$$

For that reason the solution must be written in following form in mentioned area:

$$\varphi^{(I.)}(r;z)=Y^0\left[A_2\,I_0\left(\frac{\pi}{h}r\right)+B_2\,K_0\left(\frac{\pi}{h}r\right)\right]sin\left(\frac{\pi}{h}z\right)+M_0\,z+N_0\,. \tag{35}$$

After substituting (35) to (30) we can determine integration constants:

$$N_0=\varphi_1\,,\ M_0=\frac{\varphi_2-\varphi_1}{h}=\frac{U}{h}\,. \tag{36}$$

and spatial changes of scalar electrostatic potential write as follows:

$$\varphi^{(I.)}(r;z)=Y^0\left[A_2\,I_0\left(\frac{\pi}{h}r\right)+B_2\,K_0\left(\frac{\pi}{h}r\right)\right]sin\left(\frac{\pi}{h}z\right)+\frac{U}{h}z+\varphi_1\,. \tag{37}$$

2. Spatial changes of scalar potential is determined by function $\varphi^{(II.)}(r;z)$ in area $R_2\le r\le R_1$, $z\le 0\le h$ (area II). Function $\varphi^{(II.)}(r;z)$ must obey following conditions:

$$\varphi^{(II.)}(r,0)=\varphi_1\,,\ \varphi^{(II.)}(R_2;z)=\varphi^{(I.)}(R_2;z) \tag{38}$$

The boundary conditions can valid in mentioned form in case if:

$$L_0=0\,,\ N_0=\varphi_1\,. \tag{39}$$

If we consider following substitution:

$$M_0=-K_0\,ln\,r_0\,, \tag{40}$$

where r_0 is constant, the spatial changes of scalar potential can be written:

$$\varphi^{(II.)}(r;z)=Y^0\left[A_2\,I_0\left(\frac{\pi}{h}r\right)+B_2\,K_0\left(\frac{\pi}{h}r\right)\right]sin\left(\frac{\pi}{h}z\right)+K_0\,z\,ln\left(\frac{r}{r_0}\right)+\varphi_1\,. \tag{41}$$

We assume that:

$$r>>h \tag{42}$$

in considered area, then:

$$\frac{\pi r}{h}>>1 \tag{43}$$

and Bessel`s functions arguments are very large. It holds follows for very large values of arguments *x*:

$$I_0(x)\rightarrow\frac{1}{\sqrt{2\,\pi\,x}}e^{x}\ ,\ K_0(x)\rightarrow\sqrt{\frac{\pi}{2\,x}}\,e^{-x}\ . \tag{44}$$

In addition the function $\varphi^{(II.)}(r;z)$ must be decreasing and $A_2=0$ for that reason. Consequently the spatial changes of scalar potential in dielectric media inserted between electrodes is determined by following functions:

$$\varphi^{(I.)}(r;z) = A\sqrt{\frac{h}{2r}}e^{\frac{\pi r}{h}}\sin\left(\frac{\pi}{h}z\right) + \frac{U}{h}z + \varphi_1$$ in the area $0 \le r \le R_2$, $z \le 0 \le h$ (45)

$$\varphi^{(II.)}(r;z) = A\sqrt{\frac{h}{2r}}e^{-\frac{\pi r}{h}}\sin\left(\frac{\pi}{h}z\right) + K_0 z\ln\left(\frac{r}{r_0}\right) + \varphi_1,$$ in the area $R_2 \le r \le R_1$, $z \le 0 \le h$ (46)

where A is constant. Charge surface density is the same in the both of electrodes and it is determined as follows:

$$\sigma^{(I.)} = \varepsilon E_z^{(I.)}(r;0) \quad \text{for } r \le R_2, \tag{47}$$

$$\sigma^{(II.)} = \varepsilon E_z^{(II.)}(r;0) \quad \text{for } R_2 \le r \le R_1, \tag{48}$$

where ε is dielectric constant of material inserted between electrodes and $E_z^{(I.)}$, $E_z^{(II.)}$ are components of electrostatic intensity vector oriented to direction of z coordinate in areas I and II respectively. We consider:

$$\vec{E}^{(I.)} = -\nabla\varphi^{(I.)} , \tag{49}$$

$$\vec{E}^{(II.)} = -\nabla\varphi^{(II.)} \tag{50}$$

and determine mentioned components:

$$E_z^{(I.)} = -\frac{\partial\varphi^{(I.)}}{\partial z} - A\sqrt{\frac{h}{2r}}\frac{\pi}{h}e^{-\frac{\pi}{h}r}\cos\left(\frac{\pi}{h}z\right) - \frac{U}{h} \tag{51}$$

$$E_z^{(II.)} = -\frac{\partial\varphi^{(II.)}}{\partial z} - A\sqrt{\frac{h}{2r}}\frac{\pi}{h}e^{-\frac{\pi}{h}r}\cos\left(\frac{\pi}{h}z\right) - K_0\ln\left(\frac{r}{r_0}\right) \tag{52}$$

After substituing (45) and (46) to second equation (38) we obtain:

$$K_0 = \frac{U}{h\ln\left(\frac{R_2}{r_0}\right)} . \tag{53}$$

Constant r_0 we can find by account that the amount of charge accumulated on both charged electrodes is same. It must be hold following:

$$Q = \varepsilon\int_{(S)}\sigma\, dS = \varepsilon\int_0^{R_2}\sigma^{(I.)}(r)2\pi r\, dr = \varepsilon\int_0^{R_2}\sigma^{(I.)}(r)2\pi r\, dr + \varepsilon\int_{R_2}^{R_1}\sigma^{(II.)}(r)2\pi r\, dr . \tag{54}$$

If we consider equation (54) we obtain:

$$\int_{R_2}^{R_1}\sigma^{(II.)}(r)2\pi r\, dr = 0 \tag{55}$$

and after substituing (48) and (52) to (55) we can write:

$$\frac{A\pi}{h}\sqrt{\frac{h}{2}}\int_{R_1}^{R_2}\frac{e^{-\frac{\pi}{h}r}}{\sqrt{r}}dr - 2\pi K_0\int_{R_1}^{R_2}\ln\left(\frac{r}{r_0}\right)dr = 0 . \tag{56}$$

Concerning (43) the first integral in (56) can be neglected:

$$\frac{A\pi}{h}\sqrt{\frac{h}{2}}\int_{R_1}^{R_2}\frac{e^{-\frac{\pi}{h}r}}{\sqrt{r}}dr \to 0 \tag{57}$$

and equation (56) gets the form:

$$2\pi K_0 \int_{R_1}^{R_2} ln\left(\frac{r}{r_0}\right) d\,r = 0\,. \tag{58}$$

After integrating (58) we can determine r_0 as:

$$r_0 = \frac{R_2}{\sqrt{e}}\left(\frac{R_2}{R_1}\right)^{\frac{R_1^2}{R_2^2-R_1^2}}\,. \tag{59}$$

and K_0 by substituting (59) to (53) consequently:

$$K_0 = \frac{U}{h\left\{\left(\frac{R_1^2}{R_2^2-R_1^2}\right) ln\frac{R_1}{R_2}+\frac{1}{2}\right\}} \tag{60}$$

We can find the spatial change of scalar potential in the area II. on the surface of dielectric media ($z = h$) by substituing (59) and (60) to equation (46). Mentioned potential can be evaluated under considered conditions (43) and (57) as follows:

$$U_v(r;h) = \left\{1 - k\ln\frac{r}{R_2}\right\}U \tag{61}$$

where constant k is determined by:

$$k = \frac{1}{\left(\frac{R_1^2}{R_1^2-R_2^2}\right) ln\frac{R_1}{R_2}+\frac{1}{2}} \tag{62}$$

PROBLEM OF UNCERTAINTY EVALUATION

In real cases the uncertainty of voltage applied on electrodes determination can be evaluated by:

$$\delta U \approx U - U_{v\,\min} = U - U_v(R_1;h)\,. \tag{63}$$

As it can be concluded on the basis of result (61):

$$\frac{\delta U}{U} \approx k\ln\frac{R_1}{R_2}\,. \tag{64}$$

In case if the electrodes disparity mentioned above is considered as a uncertainty in the electrodes diameter determination δR we can expect:

$$R_2 \approx R\,, R_1 - R_2 \approx \delta R\,, \tag{65}$$

Capacity of condensor is defined:

$$C = \frac{Q}{U} \tag{66}$$

and its uncertainty can be determided as follows:

$$\delta C = \frac{\partial C}{\partial U}\delta U = -\frac{Q}{U^2}\delta U\,. \tag{67}$$

consequently. As it can be easy seen from (67):

$$|\delta C| = \frac{Q}{U}\frac{\delta U}{U} = C\frac{\delta U}{U} \text{ i.e.} \left|\frac{\delta C}{C}\right| = \left|\frac{\delta U}{U}\right|\,. \tag{68}$$

If the (64), (62) and (68) are considered following relation can be written for the condensor capacity uncertainty evaluation:

$$\frac{\delta C}{C} \approx \frac{2\,\frac{\delta R}{R}\left(\frac{\delta R}{R}+2\right)\ln\left(\frac{\delta R}{R}+1\right)}{2\left(\frac{\delta R}{R}+1\right)^2 \ln\left(\frac{\delta R}{R}+1\right)+\frac{\delta R}{R}\left(\frac{\delta R}{R}+2\right)}\,. \tag{70}$$

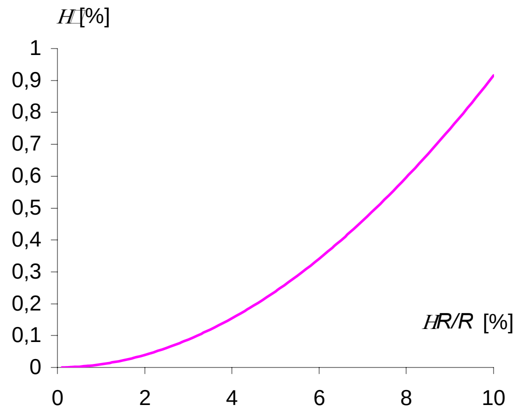
Correlation between uncertainties of determination of both capacity and condensor electrode radius is described by (70). Mentioned correlation was evaluated on the basis of scalar field dispersion phenomena analysis on the condensor electrode edges in the case if inaccuracy of determination of condensor dimensions is allowed. Uncertainty of capacity can be evaluated if the uncertainty of electrodes diameter is identified. Difference between the investigated uncertanties can be evaluated as follows:

$$\Delta = \frac{\delta R}{R} - \frac{\delta C}{C} \tag{71}$$

Tab.1

$\delta R/R$ [%]	$\delta C/C$ [%]
0,1	0,099900
1	0,990091
2	1,960721
3	2,912412
4	3,845671
5	4,760983
6	5,658823
7	6,539645
8	7,403892
9	8,251992
10	9,084357

Fig.2 Difference between electrode radius and capacity uncertainies vs. the radius uncertainty



The results are shown in tab.1 and fig.2 respectively for the values $\delta R/R$ from 0,1 % up to 10 %. That is clear the values of uncertainty of condensor radius determination is bigger in all cases then the maximal value of uncertainty of capacity caused by the field dispersion relating to the electrodes disparity.

CONCLUSION

A lot of experimental methods are based on observation of interaction of material structure with time variable electromagnetic field. Impedance spectroscopy belongs to mentioned methods.

Impedance spectroscopy is a powerful sensing tool for non-invasive observation of material structure.The impedance spectroscopy technique consists in the measurement of the sample electrical impedance as a function of frequency of the input signal over a wide frequency range. The method of impedance spectroscopy is widely used to measure electrical or dielectrical properties of materials. The usage of the mentioned method is succeful if any changes of the physical or chemical properties of material lead to changes of the electrical properties. The general approach of the method is to apply the harmonic shaped voltage as an electrical stimulus impulse to the material and observe the resulting current response. The voltage is applied over a wide frequency range. To discuss the electrical properties of sample and correlated properties as well it has to be construct a physical model of the system sample-electrodes. The results presented above can be considered in case if electrodes disparity occurs and field disspersion is expected.

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